2023/TDC(CBCS)/EVEN/SEM/
MTMHCC–602T/037

TDC (CBCS) Even Semester Exam., 2023

MATHEMATICS

(Honours)

(6th Semester)

Course No. : MTMHCC–602T

(Linear Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Prove that in a vector space $V(F)$, $0 \cdot x = 0$, $\forall\, x \in V$.

2. Let $S = \{(1, 4), (0, 3)\}$ be a subset of $\mathbb{R}^2(\mathbb{R})$. Show that $(2, 3) \in L(S)$.

3. Prove that if $V(F)$ is a vector space of dimension n, then any $n+1$ vectors in V are linearly dependent over F.

4. Examine whether the mapping $T: \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x, y, z) = x^2 + y^2 + z^2$ is a linear transformation.

5. Find the nullity of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(x, y) = (x, \ x+y, \ y)$.

6. If V is a finite dimensional vector space, prove that a linear transformation $T: V \to V$ is one-one if T is onto.

7. Define isomorphism between two vector spaces and give an example.

8. Show that inverse of a linear transformation, when it exists, is again a linear transformation.

9. Prove that a linear transformation $T: V \to W$ is non-singular if T carries each linearly independent subset of V onto a linearly independent subset of W.

10. Define eigenvalue and eigenvector of a linear operator.

11. Define eigenspace of a linear operator $T: V \to V$ associated with an eigenvalue of it and prove that it is a subspace of V.

12. Define minimal polynomial of a linear operator.

13. Let V be an inner product space. Show that $\langle u, v \rangle = 0$, for all $v \in V \Rightarrow u = 0$.

14. Using Cauchy-Schwarz inequality, prove that cosine of an angle is of absolute value at most 1.

15. Prove that an orthonormal set of non-zero vectors in an inner product space is linearly independent.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. *(a)* Prove that a necessary and sufficient condition for a non-empty subset W of a vector space $V(F)$ to be a subspace is that W is closed under vector addition and scalar multiplication. 5

(b) If S_1 and S_2 are two subsets of a vector space $V(F)$, prove that—

(i) $S_1 \subseteq S_2 \Rightarrow L(S_1) \subseteq L(S_2)$

(ii) $L(S_1 \cup S_2) = L(S_1) + L(S_2)$

(iii) $L(L(S_1)) = L(S_1)$ 1+2+2=5

17. (a) If V is a finite dimensional vector space and $\{v_1, v_2, ..., v_r\}$ is a linearly independent subset of V, then prove that $\{v_1, v_2, ..., v_r\}$ can be extended to form a basis of V. 5

(b) Define dimension of a vector space. If W is a subspace of a finite dimensional vector space $V(F)$, then prove that

$$\dim\left(\frac{V}{W}\right) = \dim V - \dim W$$ 1+4=5

18. (a) Define kernel and range of a linear transformation. If $T: V \to V$ is a linear operator, show that the following statements are equivalent : 1+1+3=5

(i) Range $(T) \cap$ Ker $(T) = \{0\}$

(ii) If $T(T(v)) = 0$, then $T(v) = 0$, $v \in V$

(b) Define rank and nullity of a linear transformation. Find the rank and nullity of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T(x, y, z) = (x+z, x+y+2z, 2x+y+3z)$$ 1+1+3=5

19. (a) State and prove Sylvester's law of nullity. 5

(b) Define matrix of a linear transformation. Find the matrix of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x+y, 2z-x)$$

with respect to the standard ordered basis of $\mathbb{R}^3$ and $\mathbb{R}^2$. 2+3=5

20. (a) Let $U(F)$ and $V(F)$ be two vector spaces and $T: V \to U$ be a linear transformation. Prove that

$$\frac{V}{\text{Ker } T} \cong \text{Range } T$$ 5

(b) Let V and W be two vector spaces over a field F of dimensions m and n respectively. Prove that Hom (V, W) has dimension mn, where Hom (V, W) is the vector space of all linear transformations from V to W. 5

21. (a) If A and B are two subspaces of a vector space $V(F)$, then prove that

$$\frac{A+B}{A} \cong \frac{B}{A \cap B}$$ 5

(b) If $T_1,\ T_2 \in \text{Hom}\ (V,\ W)$, then show that

(i) $r(\alpha T_1) = r(T_1)$ for all $\alpha \in F,\ \alpha \neq 0$

(ii) $|r(T_1) - r(T_2)| \leq r(T_1 + T_2) \leq r(T_1) + r(T_2)$

where $r(T)$ means rank of T. 2+3=5

22. (a) Let T be a linear operator on a finite dimensional vector space V over a field F. Prove that $c \in F$ is an eigenvalue of T if and only if $T - cI$ is singular. 5

(b) State and prove Cayley-Hamilton theorem. 5

23. (a) Let V be a finite dimensional vector space over the field $\mathbb{R}$ of real numbers and $\dim V = 2$. Let T be a linear operator on V such that $T(v_1) = \alpha v_1 + \beta v_2$, $T(v_2) = \gamma v_1 + \delta v_2$, where $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$ and $\{v_1,\ v_2\}$ is a basis of V. Find necessary and sufficient condition that 0 is an eigenvalue of T. 5

(b) Determine the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ 5

24. (a) Let V be an inner product space. Prove that $|\langle u,\ v\rangle| \leq \|u\|\|v\|$, for all $u,\ v \in V$. Also, prove that $|\langle u,\ v\rangle| \leq \|u\|\|v\|$ if and only if u and v are linearly dependent. 3+2=5

(b) Let v be a non-zero inner product space of dimension n. Prove that V has an orthonormal basis. 5

25. (a) State and prove Bessel's inequality. 5

(b) Let W_1 and W_2 be subspaces of a finite dimensional inner product space V. Show that

(i) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

(ii) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$ 2½+2½=5

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