2023/TDC(CBCS)/EVEN/SEM/ MTMHCC–201T/028

TDC (CBCS) Even Semester Exam., 2023

MATHEMATICS

(Honours)

(2nd Semester)

Course No. : MTMHCC–201T

(Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* questions : 2×10=20

1. Show that $\mathbb{R}$ is not bounded above in $\mathbb{R}$.

2. Show that

$$\bigcap_{n=1}^{\infty} [n, \infty) = \varnothing$$

3. Assuming density of $\mathbb{Q}$ in $\mathbb{R}$, show that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

4. Show that 5 is not a limit point of $\mathbb{N}$.

5. Show that [0, 1) is not open in $\mathbb{R}$.

6. Does the set $(-5, 0) \cup \mathbb{N}$ have a limit point in $\mathbb{R}$? Justify.

7. Show that the sequence (x_n), where

$$x_n = (-1)^{n+1} \quad \forall\, n \in \mathbb{N}$$

cannot converge to 1.

8. Find a sequence of irrationals (x_n) such that $x_n \to 0$.

9. Let the sequence (x_n) converges to $x_0 \in \mathbb{R}$. Find an upper bound of the set

$$\{x_n : n \in \mathbb{N}\}$$

10. Prove or disprove :

$$\left(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{7}, \cdots\right)$$

is a subsequence of the sequence $\left(\frac{1}{n}\right)$.

11. Prove or disprove : Every Cauchy sequence in $\mathbb{R}$ is monotone.

12. Write down four convergent subsequences of the sequence (x_n), where

$$x_n = (-1)^n \quad \forall\, n \in \mathbb{N}$$

13. Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series such that $x_n < y_n \quad \forall\, n \in \mathbb{N}$. Show that

$$\sum_{n=1}^{\infty} x_n < \sum_{n=1}^{\infty} y_n$$

14. Prove or disprove : If the series $\sum_{n=1}^{\infty} x_n$ converges, then $x_n \to 0$.

15. Show that the series $\sum_{n=1}^{\infty} \frac{1}{\underline{|n}}$ converges.

SECTION—B

Answer any *five* questions : 10×5=50

16. *(a)* Show that $\mathbb{Z}$ is a countable set. 5

(b) Let A and B be non-empty bounded below subsets of $\mathbb{R}$. Show that

$$\text{g.l.b.}(A+B) = \text{g.l.b.}(A) + \text{g.l.b.}(B) \qquad 5$$

17. *(a)* Show that the following are equivalent : 5

(i) $\mathbb{N}$ is unbounded above in $\mathbb{R}$

(ii) $\forall\, x, y \in \mathbb{R}$ with $x > 0$, $\exists\, n \in \mathbb{N}$ such that $nx > y$

(b) Assuming order-completeness of $\mathbb{R}$, show that every non-empty bounded below subset of $\mathbb{R}$ has an infimum. 5

18. *(a)* Show that $A \subset \mathbb{R}$ is open in $\mathbb{R}$ iff $\mathbb{R} \setminus A$ contains all its limit points. 5

(b) Let A be a non-empty bounded above open subset of $\mathbb{R}$. Show that

$$\text{l.u.b.}(A) \notin A$$ 5

19. *(a)* Show that $\mathbb{Q}$ is not closed in $\mathbb{R}$. Find $\overline{\mathbb{Q}}$ in $\mathbb{R}$. 3+2=5

(b) Find the derived set of $\mathbb{R} \setminus \{0\}$ in $\mathbb{R}$. Is $\mathbb{R} \setminus \{0\}$ closed? Justify. Is $\mathbb{R} \setminus \{0\}$ open? Justify. (No credit will be given without proper justification) 2+1+2=5

20. *(a)* Let (x_n) converges to $x_0 \in \mathbb{R}$. Show that

$$\left\{ n \in \mathbb{N} : |x_n - x_0| \geq \frac{1}{2} \right\}$$

is a finite set. 5

(b) Prove or disprove : If (x_n) and (y_n) both do not converge, then $(x_n y_n)$ and $(x_n + y_n)$ both cannot converge. 5

21. *(a)* Let (x_n) be a monotone decreasing bounded sequence. Show that (x_n) converges. 5

(b) Let (x_n) be a bounded sequence and (y_n) converges to 0. Show that $(x_n y_n)$ converges to 0. Does $(x_n y_n)$ converge if (x_n) is bounded and $y_n \to 1$? 3+2=5

22. *(a)* Prove or disprove : There exists a bounded sequence (x_n) in $\mathbb{R}$ such that given any subsequence (x_{n_k}) of (x_n) and given any $x_0 \in \mathbb{R}$, (x_{n_k}) does not converge to x_0. 5

(b) Show that the sequence (x_n), where

$$x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad \forall\, n \geq 3$$

is not a Cauchy sequence in $\mathbb{R}$. 5

23. *(a)* Let (y_n) and (z_n) be subsequences of (x_n) such that $y_n \to y_0$ and $z_n \to z_0$ in $\mathbb{R}$. If $y_0 \neq z_0$, show that (x_n) is not convergent. 5

(b) Let (x_n) be a Cauchy sequence in $\mathbb{R}$ such that $x_n \in \mathbb{N}$ $\forall\, n \in \mathbb{N}$. Show that there exists $m \in \mathbb{N}$ such that

$$x_n = x_k \quad \forall\, n, k \geq m$$ 5

24. *(a)* Show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$. 5

(b) If $\sum_{n=1}^{\infty} x_n$ with $x_n > 0 \ \forall n \in \mathbb{N}$, is convergent, then is $\sum_{n=1}^{\infty} x_n^2$ always convergent? Justify your answer. 5

25. *(a)* Let $\sum_{n=1}^{\infty} x_n$ be a convergent series such that $\sum_{n=1}^{\infty} x_n y_n$ is convergent. Is $\sum_{n=1}^{\infty} y_n$ necessarily convergent? Justify. Is $\sum_{n=1}^{\infty} y_n$ necessarily divergent? Justify. 4

(b) Let (x_n) be a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} | x_n |$$

converges. Show that $\sum_{n=1}^{\infty} x_n$ is convergent. Does convergence of $\sum_{n=1}^{\infty} x_n$ necessarily imply the convergence of $\sum_{n=1}^{\infty} | x_n |$? Justify. 3+3=6

★ ★ ★