2022/TDC(CBCS)/EVEN/SEM/ MTMSEC–401T (A/B/C)/263

TDC (CBCS) Even Semester Exam., 2022

MATHEMATICS

(4th Semester)

Course No. : MTMSEC–401T

Full Marks : 50
Pass Marks : 20

Time : 3 hours

The figures in the margin indicate full marks for the questions

Candidates have to answer *either* from Option A or Option B or Option C

OPTION—A

Course No. : MTMSEC–401T (A)

(Graph Theory)

SECTION—A

Answer any *fifteen* of the following questions : 1×15=15

1. Define a graph.

2. Define degree of a vertex of a graph.

3. What is a bipartite graph?

4. What is the maximum number of edges in a simple graph with n vertices?

5. What is a tree?

6. What is a leaf in a tree?

7. What is a spanning tree of a connected graph?

8. How many edges does a tree with n vertices have?

9. Define isomorphism from a graph to another graph.

10. What is an Eulerian circuit?

11. What is a Hamiltonian cycle?

12. Define adjacency matrix of a graph.

13. What is a planar graph?

14. Give example of a graph that is not planar.

15. State the necessary and sufficient conditions for a graph to be planar.

16. For which n, is K_n planar?

17. What is the travelling salesman's problem?

18. What is a weighted graph?

19. Name an algorithm to find the shortest path from a vertex to another vertex in a weighted graph.

20. What is Floyd-Warshall algorithm used for?

SECTION—B

Answer any *five* of the following questions : $2\times5=10$

21. Define subgraph of a graph and illustrate it with diagrams.

22. Show that the sum of the degrees of the vertices of a graph is equal to twice the number of edges.

23. What is a bridge in a connected graph? Illustrate with a diagram.

24. Draw all possible labelled trees on three vertices.

25. Prove that any circuit in a graph must contain a cycle.

26. Write the adjacency matrix of the complete bipartite graph $K_{2,3}$.

27. Show that K_5 is not planar.

28. Show that every planar graph has at least one vertex of degree $d \leq 5$.

29. Write the steps of Dijkstra's algorithm.

30. Write the steps of Floyd-Warshall algorithm.

SECTION—C

Answer any *five* of the following questions : 5×5=25

31. Draw the graphs K_5 and $K_{3,4}$. 2+3=5

32. Suppose all the vertices in a graph have odd degree K. Show that the total number of edges in the graph is a multiple of K.

33. Show that a connected graph with n vertices is a tree if and only if it has $n-1$ edges.

34. Prove that a graph is a tree if and only if it is connected and every edge is a bridge.

35. Show that isomorphism is an equivalence relation on the set of all graphs.

36. Show that the number of walks of length 2 in any graph G is the sum of the entries of the matrix A^2 where A is the adjacency matrix of G.

37. Let G be a planar graph with $V \geq 3$ vertices and E edges. Show that $E \leq 3V - 6$.

38. Let G be a connected graph with V_1 vertices and E_1 edges and let H be a subgraph with V_2 vertices and E_2 edges. Show that

$$E_2 - V_2 \leq E_1 - V_1$$

39. Explain the improved version of Dijkstra's algorithm.

40. Illustrate Floyd-Warshall algorithm with an example.

OPTION—B

Course No. : MTMSEC–401T (B)

(Special Function)

SECTION—A

Answer any *fifteen* of the following questions : 1×15=15

1. Write down the Legendre's polynomial of first kind of order n.

2. What are the roots of $P_n(x) = 0$, where $P_n(x)$ is a Legendre's polynomial of first kind?

3. When n is positive integer, the value of
$$\frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{[x \pm \sqrt{(x^2-1)}\cos\phi]^{n+1}}$$
is ____. (Fill in the blank)

4. Write down the Legendre's polynomial of second kind of order n.

5. Write down the value of
$$\int_{-1}^{1} P_m(x)P_n(x)\,dx$$
when $m = n$.

6. Write the Bessel's function of first kind of order n.

7. Expand $J_n(x)$ in the powers of x when $n = 0$.

8. If
$$J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$$
express $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

9. Define Laplace transform of a function $f(x)$.

10. If $f(x) = \sinh ax$, what is $L(f(x))$, where $L(f(x))$ is Laplace transform of $f(x)$?

11. What is the inverse Laplace transform of $\frac{1}{s^2+a^2}$?

12. Prove that Laplace transform of 1 is $\frac{1}{s}$, i.e., $L(1) = \frac{1}{s}$.

13. If $L\{F(t)\} = f(s)$, then what is $L(F'(t))$, where ($'$) denotes differentiation w.r.t. t?

14. If $L(F(t)) = f(s)$, then what is the value of $L(F''(t))$?

15. If $L(F(t)) = f(s)$, then what is the value of $L\{F^n(t)\}$?

16. Write down the value of $L(t^n F(t))$, if $L(F(t)) = f(s)$.

17. Write down the formula for infinite Fourier sine transformation of $f(x)$.

18. What is the formula for infinite Fourier transformation of $f(x)$?

19. Write Fourier cosine integral formula.

20. Write Fourier exponential integral formula.

SECTION—B

Answer any *five* of the following questions : 2×5=10

21. Show that $P_n(1) = 1$.

22. Prove that $P_{2m+1}(0) = 0$.

23. Using Rodrigue's formula, prove that

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

24. Prove that

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

25. Find the Laplace transform of $\sin 2t \cdot \sin 3t$.

26. Find the inverse Laplace transform of

$$\left(\frac{s^2 - 3s + 4}{s^3}\right)$$

27. Using Laplace transformation, find $L(\cos 2t)$.

28. Using Laplace transformation, find $L\{\sin(nt + \alpha)\}$.

29. If

$$f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x \geq a \end{cases}$$

then find Fourier cosine transformation of $f(x)$.

30. If $f(x) = 2e^{-5x} + 5e^{-2x}$, then find Fourier sine transformation of $f(x)$.

SECTION—C

Answer any *five* of the following questions : 5×5=25

31. When n is positive integer, then prove that

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}[x \pm \sqrt{(x^2-1)}\cos\theta]^n \, d\theta$$

Hence find $P_n(\cos\phi)$.

32. Show that

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Hence find the value of $P_n(-1)$.

33. Prove that

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}(x^2-1)^n$$

34. Show that if n is a positive integer

$$J_{-n}(x) = (-1)^n J_n(x)$$

Also show that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$$

2+3=5

35. *(a)* Find Laplace transform of

$$e^{-3t}(2\cos 5t - 3\sin 5t)$$

2

(b) Find inverse Laplace transform of

$$\frac{s+2}{s^2-4s+13}$$

3

36. *(a)* Find inverse Laplace transform of

$$\frac{4s+5}{(s-1)^2(s+2)}$$

2

(b) Given the function

$$f(x) = \begin{cases} 2, & 0 \le x < 1 \\ 2x, & x \ge 1 \end{cases}$$

Find $L(f(x))$. 3

37. Solve the differential equation

$$(D^2+9)y = \cos 2t$$

if $y(0)=1$, $y'(\pi/2) = -1$.

38. Solve the differential equation

$$(D^2+n^2)x = a\sin(nt+\alpha)$$

if $x(0)=0$, $x'(0)=0$.

39. Using Fourier transforms, show that

$$\int_0^{\infty} \frac{\cos sx}{s^2+1} ds = \frac{\pi}{2} e^{-x} (x \ge 0)$$

40. Using Fourier integral, show that

$$e^{-ax} = \frac{2a}{\pi}\int_0^{\infty} \frac{\cos\lambda x}{\lambda^2+a^2} d\lambda, \; a>0, \; x \ge 0$$

OPTION—C

Course No. : MTMSEC–401T (C)

(Vector Analysis/Vector Calculus)

SECTION—A

Answer any *fifteen* of the following questions : 1×15=15

1. Determine the value of λ, for which the vectors $\vec{a} = \lambda\hat{i} - 4\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + \lambda\hat{j} - 2\hat{k}$ are perpendicular.

2. What is the condition for coplanarity of three vectors?

3. Find the value of
$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$$

4. Write the vector equation of the line passing through the point $\hat{i} - \hat{j} + \hat{k}$ and parallel to the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

5. Define vector function of a scalar variable.

6. Let $\vec{u}(t)$ and, $\vec{v}(t)$ are differentiable vector functions of a scalar variable t. Then what is the value of $\frac{d}{dt}(\vec{u} \times \vec{v})$?

7. Prove that
$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{r} \times \frac{d^2\vec{r}}{dt^2}$$

8. What is the necessary and sufficient condition for a vector function $\vec{r} = \vec{f}(t)$ of a scalar variable t in a domain $D \subseteq R$, to have a constant magnitude?

9. What is the gradient of a constant function?

10. If $u = x^3 + 3yz^2$, then find $\vec{\nabla} u$.

11. Define the curl of a vector point function.

12. Define irrotational vector.

13. Write the value of
$$\int\left(\vec{r} \cdot \frac{d\vec{A}}{dt} + \vec{A} \cdot \frac{d\vec{r}}{dt}\right) dt$$

14. Write the value of

$$\int \left(2\vec{r} \cdot \frac{d\vec{r}}{dt} \right) dt$$

15. If $\vec{a}(u) = u^2\hat{i} + (u-1)\hat{j} - 4\hat{k}$, then find $\int \vec{a}(u)\, du$.

16. Write the value of

$$\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2} \right) dt$$

17. State the principle of work.

18. What is the principle of conservation of linear momentum?

19. Define kinetic energy. Is it a scalar or vector quantity?

20. A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity of the particle at time $t = 0$.

SECTION—B

Answer any *five* of the following questions : 2×5=10

21. Find the perpendicular distance of the plane $\vec{r} \cdot (2\hat{i} - 2\hat{j} + \hat{k}) + 6 = 0$ from the origin.

22. For any non-zero non-coplanar vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, show that

$$[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}] = 2[\vec{a},\ \vec{b},\ \vec{c}]$$

23. If $\vec{a} = \hat{i} - t^2\hat{j} + t^3\hat{k}$ and $\vec{b} = (\sin t)\hat{i} + (\cos t)\hat{j}$, then find the value of $\frac{d}{dt}(\vec{a} \cdot \vec{b})$.

24. Define continuity of a vector function of a scalar variable.

25. Prove that curl grad $\phi = \vec{0}$.

26. Prove that div curl $\vec{f} = 0$.

27. Evaluate

$$\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2} \right) dt$$

where $\vec{r} = 2t^2\hat{i} + t\hat{j} - 3t^2\hat{k}$.

28. If $\vec{v}$ be a vector function of a scalar variable t and $\frac{d\vec{v}}{dt} = e^t\hat{i} + e^{2t}\hat{j} + \hat{k}$, then find $\vec{v}$; given that $\vec{v} = \hat{i} + \hat{j}$ when $t = 0$.

29. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 3\sin 3t$, where t is the time. Find the velocity and acceleration of the particle at any time t.

30. Find the work done by the force $\vec{F} = (0, 0, -mg)$ in moving a particle of mass m from $O(0, 0, 0)$ to $A(1, 1, 1)$ along the curve $x = t, y = t^2, z = t^3$; t being a parameter.

SECTION—C

Answer any *five* of the following questions : 5×5=25

31. *(a)* Prove that

$$[\vec{a}, \vec{b}, \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} & \vec{c}\cdot\vec{c} \end{vmatrix}$$ 3

(b) Find the vector equation of the sphere whose centre is $\vec{c} = 2\hat{i} - \hat{j} + \hat{k}$ and radius is 5 units. 2

32. *(a)* Prove that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$ 3

(b) If the position vectors of the vertices of a tetrahedron are $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 4\hat{j} + 3\hat{k}$, $3\hat{i} + 2\hat{j} + 4\hat{k}$ and $4\hat{i} + 3\hat{j} + 2\hat{k}$, then find its volume. 2

33. *(a)* If $\vec{r} = \vec{f}(t)$ is a differentiable vector function of a scalar variable t, then write the geometrical interpretation of $\frac{d\vec{r}}{dt}$. 2

(b) Find the unit tangent vector at any point to the curve $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$. 3

34. *(a)* If $\vec{r} = (a\cos t)\hat{i} + (a\sin t)\hat{j} + (at\tan\alpha)\hat{k}$, then find $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right]$. 3

(b) If $\vec{r} = (\cos t)\hat{i} + (\sin t)\hat{j}$, prove that $\frac{d\vec{r}}{dt}$ is perpendicular to $\vec{r}$. 2

35. *(a)* Prove that

$$\text{curl curl } \vec{f} = \text{grad div } \vec{f} - \nabla^2 \vec{f}$$ 3

(b) If $\phi = 2x^3y^2z^4$, then find div (grad ϕ). 2

36. *(a)* If $\vec{F} = (x+y+1)\hat{i} + \hat{j} + (-x-y)\hat{k}$, then prove that $\vec{F} \cdot \text{curl}\, \vec{F} = 0$. 3

(b) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $\text{div}(\vec{r} \times \vec{a})$ where $\vec{a}$ is a constant vector. 2

37. *(a)* If $\vec{f}(t) = t\hat{i} + (t^2 - 2t)\hat{j} + (3t^2 + 3t^3)\hat{k}$, then find $\int_0^1 \vec{f}(t)\, dt$. 3

(b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = x^2y^2\hat{i} + y\hat{j}$ and the curve C is $y^2 = 4x$ in the xy-plane from $(0, 0)$ to $(4, 4)$. 2

38. *(a)* Show that

$$\int_0^{\frac{\pi}{2}} (5\cos u\hat{i} - 7\sin u\hat{j})\, du = 5\hat{i} - 7\hat{j}$$ 2

(b) If $\vec{A} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$, then evaluate $\int \vec{A} \cdot d\vec{r}$ along the curve C given by $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$. 3

39. *(a)* A particle moves according to the law $\vec{r} = (\cos t)\hat{i} + (\sin t)\hat{j} + t^2\hat{k}$. Find the magnitude of the tangential and the normal components of acceleration. 3

(b) The acceleration of a particle at any time t is $e^t\hat{i} + e^{2t}\hat{j} + \hat{k}$. Find the velocity at time t, if the initial velocity be $\hat{i} + \frac{1}{2}\hat{j}$. 2

40. *(a)* Find the work done by the force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ in moving a particle in the xy-plane from $(0, 0)$ to $(1, 1)$ along the parabola $y^2 = x$. 3

(b) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$. Find the components of the velocity in the direction $\hat{i} - 3\hat{j} + 2\hat{k}$ at time $t = 1$. 2

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