TDC (CBCS) Even Semester Exam., 2022

MATHEMATICS

(Honours)

(4th Semester)

Course No. : MTMHCC–403T

(Ring Theory)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Give example, with brief justification, of a non-commutative ring with unity.

2. Let R be a ring and $a \in R$. Show that

$$a \cdot 0 = 0 \cdot a = 0$$

where 0 is the additive identity in R.

3. Define integral domain. Give one example of an integral domain.

4. Define left ideal and right ideal of a ring.

5. If an ideal I of a ring R contains a unit, show that $I = R$.

6. A is a prime ideal and B is a maximal ideal of a commutative ring R with unity. What can you conclude about the factor rings R/A and R/B?

7. Define ring homomorphism.

8. Let $\phi: R \to S$ be a ring homomorphism. Show that if R is commutative, then $\phi(R)$ is commutative.

9. Define kernel of a ring homomorphism. Is the kernel of any ring homomorphism always non-empty? Justify.

10. Consider the elements $f(x) = 2x^3 + x^2 + x + 2$ and $g(x) = 2x^2 + 2x + 1$ in the polynomial ring $Z_3[x]$. Compute $f(x) + g(x)$ and $f(x) \cdot g(x)$.

11. State division algorithm in polynomial ring over a field.

12. Define Euclidean domain.

13. Define irreducible polynomial in $D[x]$ where D is an integral domain.

14. Give example, with justification, of a polynomial that is irreducible over Q but reducible over $\mathbb{R}$.

15. State Eisenstein's criterion of irreducibility.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. (a) Show that the set of all even integers under ordinary addition and multiplication is a commutative ring. Does it have unity? Justify. 4+1=5

(b) Show that a non-empty subset S of a ring R is a sub-ring if S is closed under subtraction and multiplication, i.e., if $a - b \in R$ and $ab \in R$ wherever $a, b \in R$. 5

17. (a) Show that every finite integral domain is a field. Give example of an integral domain that is not a field. 4+1=5

(b) Show that the nilpotent elements of a commutative ring form a sub-ring. Is the result true for non-commutative ring? 4+1=5

18. *(a)* If A and B are ideals of a ring R, show that the sum $A+B=\{a+b \mid a\in A, b\in B\}$ is also an ideal of R. 5

(b) Let R be a commutative ring with unity and A be an ideal of R. Show that R/A is an integral domain if and only if A is prime. 5

19. *(a)* Let R be a commutative ring with unity and let $a\in R$. Show that the set $\langle a\rangle := \{ra \mid r\in R\}$ is an ideal of R. 5

(b) If R is a commutative ring with unity and A is an ideal of R, show that R/A is a commutative ring with unity. Under what condition on A will R/A be a field? 4+1=5

20. *(a)* Let $\phi: R\to S$ be a ring homomorphism. Show that ϕ is an isomorphism if and only if ϕ is onto and $\ker\phi=\{0\}$. 5

(b) Let $f: R\to S$ be a ring homomorphism. Show that $R/\ker f$ is isomorphic to $f(R)$. 5

21. *(a)* Show that the kernel of a ring homomorphism is an ideal of the domain ring. 4

(b) Let A and B be ideals of a ring R with $B\subseteq A$. Show that A/B is an ideal of R/B and $(R/B)/(R/A)$ is isomorphic to R/A. 6

22. *(a)* Show that every Euclidean domain is a principal ideal domain. 5

(b) Show that in a principal ideal domain, an element is irreducible if and only if it is prime. 5

23. *(a)* If F is a field, show that $F[x]$ is a principal ideal domain. 5

(b) Let D be an integral domain. Show that the relation $a\sim b$ iff a and b are associates, is an equivalence relation on D. 5

24. *(a)* Show that in an integral domain, the product of an irreducible and a unit is irreducible. 5

(b) Show that the ring

$$\mathbb{Z}[\sqrt{-5}]=\{a+b\sqrt{-5} \mid a, b\in\mathbb{Z}\}$$

is not a UFD. 5

25. *(a)* For any prime p, show that the p-th cyclotomic polynomial

$$\phi_p(x)=\frac{x^p-1}{x-1}=x^{p-1}+x^{p-2}+\cdots+x+1$$

is irreducible over Q. 5

(b) Let F be a field and let $a\in F$ be a non-zero element. If $f(x+a)$ is irreducible over F, prove that $f(x)$ is irreducible over F. 5

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