2022/TDC(CBCS)/EVEN/SEM/ MTMHCC–201T/256

TDC (CBCS) Even Semester Exam., 2022

MATHEMATICS

(Honours)

(2nd Semester)

Course No. : MTMHCC–201T

(Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* questions : 2×10=20

1. If M is a neighbourhood of a point x and $N \supset M$, then show that N is also a neighbourhood of x.

2. Show that $\mathbb{N} \times \mathbb{N}$ is countable.

3. Let

$$S = \left\{ 1 + \frac{(-1)^n}{n},\ n \in \mathbb{N} \right\}$$

Find supremum and infimum of S.

4. If x is a limit point of a set $S \subset \mathbb{R}$, is x a limit point of $S-\{x\}$? Justify your answer.

5. Define open and closed sets.

6. Obtain the derived set of the following sets :

(i) $\{1, 2, 3, 4, 5\}$

(ii) $\{1, 2, 3, 4, \dots 500\}$

7. Find the limit of the sequence $\{x_n\}$ if

$$x_n = \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}$$

8. Give examples of divergent sequences $\{x_n\}$ and $\{y_n\}$ such that $\{x_n + y_n\}$ converges.

9. Prove that the limit of a convergent sequence is unique.

10. Define subsequence and give example.

11. Show that every convergent sequence is a Cauchy sequence.

12. State Bolzano-Weierstrass theorem for sequence.

13. Prove that if a series converges, its nth term must necessarily approach to zero.

14. Define alternating series and state when an alternating series is said to be convergent.

15. Find nth partial sum of the series

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots \infty$$

SECTION—B

Answer any *five* questions : 10×5=50

16. *(a)* Prove that every open interval in $\mathbb{R}$ is a neighbourhood of each of its points. 2

(b) Show that the countable union of countable sets is countable. 4

(c) State and prove Archimedean property. 4

17. *(a)* Show that every superset of an uncountable set is uncountable. 2

(b) Let A and B be two non-empty subsets of $\mathbb{R}$ and let

$$C = \{x + y \mid x \in A \text{ and } y \in B\}$$

Show that $\sup C = \sup A + \sup B$. 4

(c) Prove that the set of rational numbers is not order complete. 4

18. *(a)* If A and B are sets of real numbers, then show that

$$(A \cup B)' = A' \cap B'$$ 3

(b) Show that union of two open sets is an open set. 3

(c) Show that a set is closed if and only if it contains all its limit points. 4

19. *(a)* If A and B are sets of real numbers and $A \subset B$, then show that $A' \subset B'$. 2

(b) Prove that a set is closed if and only if its complement is open. 4

(c) State and prove Bolzano-Weierstrass theorem. 4

20. *(a)* Prove that the sequence with nth term

$$x_n = \frac{2n-7}{3n+2}$$

is monotonically increasing and bounded. 3

(b) If the sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively, then show that the sequence $\{x_n y_n\}$ is convergent and converges to xy. 4

(c) Show that the sequence $\{x_n\}$, where $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ $\forall\, n \geq 1$ is convergent. 3

21. *(a)* Prove that every monotonically increasing sequence bounded above is convergent and converges to its supremum. 3

(b) Prove that the sequence

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}$$

is convergent and find its limit. 4

(c) Show that the sequence $\{x_n\}$ is convergent where

$$x_n = 2 + (-1)^n \frac{1}{n}$$ 3

22. *(a)* Find subsequences of the sequence

$$\left\{\frac{n+1}{n+2}\right\}$$ 3

(b) Show that the sequence $\{x_n\}$ defined by

$$x_n = \frac{n}{n+1}$$

is a Cauchy sequence. 3

(c) State and prove Cauchy's general principle of convergence for sequence. 4

23. *(a)* Prove that the sequence $\{x_n\}$ converges to the limit l if and only if every subsequence of $\{x_n\}$ converges to l. 4

(b) Prove that every Cauchy sequence is bounded. 3

(c) Using Cauchy's general principle of convergence, show that the sequence $\{x_n\}$ where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not convergent. 3

24. *(a)* Test the convergence of the following series : 3+3=6

(i) $$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots \infty$$

(ii) $$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 2}{3\cdot 5}\right)^2 + \left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2 + \cdots$$

(b) Show that the series

$$\Sigma(-1)^{n-1}\{\sqrt{n^2+1} - n\}$$

is conditionally convergent. 4

25. *(a)* State and prove ratio test. 4

(b) Test the convergence of the following series :

$$\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots$$ 3

(c) Discuss the convergence of the series

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$$ 3

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