2021/TDC/CBCS/ODD/
MATDSE–502T(A/B)/332

TDC (CBCS) Odd Semester Exam., 2021
held in March, 2022

MATHEMATICS

(5th Semester)

Course No. : MATDSE–502T

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Candidates have to answer *either* Option—A *or* Option—B

OPTION—A

Course No. : MATDSE–502T (A)

(Analytical Geometry)

SECTION—A

Answer any *twenty* of the following questions as directed : 1×20=20

1. Under what condition the equation $ax^2 + 2hxy + by^2 = 0$ will represent a pair of parallel straight lines?

2. What will be the two invariants when the expression $ax^2 + 2hxy + by^2$ changes to $a'x'^2 + 2h'x'y' + b'y'^2$ by an orthogonal transformation?

3. Write the transformed equation of $x^2 - y^2 = 0$ when the origin is shifted to $(-1, 2)$.

4. Write the equations of the lines represented by $x^2 - 5xy - 6y^2 = 0$.

5. Write down the equations to the bisectors of the angles between the lines represented by the equation $x^2 + 2y^2 + 4xy = 0$.

6. Define radical axis in case of two circles.

7. State the condition under which the line $y = mx + c$ touches the circle $x^2 + y^2 = a^2$.

8. Find the point of contact if $y = mx + \frac{a}{m}$ be a tangent to the parabola $y^2 = 4ax$.

9. Under what condition the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ cut each other orthogonally?

10. Define radical centre of three circles.

11. Write the polar equation of a conic.

12. Define pole and polar for a given curve.

13. What is the equation of polar of the point (α, β) w.r.t. the parabola $y^2 = 4ax$?

14. Find the nature of the conic $\frac{8}{r} = 4 - 5\cos\theta$.

15. Find the polar equation of the parabola whose latus rectum is 8.

16. Write the equation of a plane in normal form.

17. If two lines are coplanar, what is the shortest distance between the lines?

18. What are the centre and radius of the sphere $x^2 + y^2 + z^2 + 2x - 4y + 2z - 3 = 0$?

19. Write down the equation of the sphere with (x_1, y_1, z_1) and (x_2, y_2, z_2) as the end points of a diameter.

20. Define great circle.

21. Define a right circular cone.

22. What should be the values of l, m, n if the generators are parallel to X-axis?

23. Every _____ degree of homogeneous equation in x, y, z represents a cone with vertex at the _____. (Fill in the blanks)

24. Define cylinder.

25. What do you mean by right circular cylinder?

SECTION—B

Answer any *five* of the following questions : 2×5=10

26. Find the value of k for which the equation $2x^2+3xy-2y^2+7x+y+k=0$ represents a pair of straight lines.

27. If the equation

$$ax^2+3xy-2y^2-5x+5y+c=0$$

represents two straight lines perpendicular to each other, then find a and c.

28. Find the radical axis of the two circles $x^2+y^2+2x+4y-7=0$ and $x^2+y^2-6x+2y-5=0$

29. Show that the line $x+2y-4=0$ touches the ellipse $3x^2+4y^2=12$.

30. Find the equation of the polar of the point (2, 3) w.r.t. the circle $x^2+y^2-2x-4y+1=0$.

31. Find the point on the conic $\frac{15}{r}=1-4\cos\theta$ whose radius vector is 5.

32. Prove that the straight lines $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ and $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ intersect each other.

33. Find the equation of the sphere passing through the origin and makes intercepts a, b, c on the coordinate axes.

34. Find the equation of the cone whose vertex is origin and guiding curve is $x^2+y^2+z^2=1, 2x+3y+4z=7$.

35. What do you mean by guiding curve and generator of a cylinder?

SECTION—C

Answer any *five* of the following questions : 8×5=40

36. *(a)* If

$$ax^2+2hxy+by^2=1 \text{ and } a'x^2+2h'xy+b'y^2=1$$

represent the same conic, axes being rectangular, then show that $(a-b)^2+4h^2=(a'-b')^2+4h'^2$. 4

(b) Find the equation of the pair lines through origin and perpendicular to the pair $ax^2+2hxy+by^2=0$. 4

37. *(a)* Show that the two straight lines through the origin which make angles 45° with the line $lx+my+n=0$ are given by

$$(l^2-m^2)(x^2-y^2)+4lmxy=0$$ 4

(b) If two pairs of lines $x^2-2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each pair bisects the angles between the other pair, then prove that $pq+1=0$. 4

38. *(a)* Find the equation of the circle which passes through the origin and cuts orthogonally each of the circles

$$x^2+y^2-6x+8=0 \text{ and } x^2+y^2-2x-2y=7$$ 4

(b) Prove that the locus of the point of intersection of two tangents to an ellipse at right angles to one another is a circle. 4

39. *(a)* Find the radical centre of the circles

$$x^2+y^2+x+2y+3=0,$$

$$x^2+y^2+2x+4y+5=0 \text{ and}$$

$$x^2+y^2-7x-8y-9=0$$ 3

(b) Prove that two tangents can be drawn from a point to a parabola and if these two tangents be perpendicular to each other, the locus of their point of intersection is the directrix. 5

40. *(a)* Find the equation of the polar of the point (x_1, y_1) w.r.t. the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. 4

(b) If PSP' and QSQ' be two perpendicular focal chords of a conic $\frac{l}{r}=1+e\cos\theta$, then show that $\frac{1}{PP'}+\frac{1}{QQ'}=$ constant. 4

41. *(a)* Find the pole of the straight line $2x - y = 6$ w.r.t. the circle $5x^2 + 5y^2 = 9$. 4

(b) Find the condition that the line $\frac{l}{r} = A\cos\theta + B\sin\theta$ may be a tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$. 4

42. *(a)* Find the length and the equations of the shortest distance between the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$ 4

(b) A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$. 4

43. *(a)* Prove that the shortest distance between the y-axis and the line

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$$

is

$$\frac{bd' - b'd}{\sqrt{(ba' - b'a)^2 + (bc' - b'c)^2}}$$ 4

(b) Show that $2x - 6y + 3z - 49 = 0$ is a tangent plane to the sphere $x^2 + y^2 + z^2 = 9$. Find the point of contact. 4

44. *(a)* Prove that the equation of the right circular cone, whose vertex is the origin, axis is OX and semi-vertical angle α, is $y^2 + z^2 = x^2 \tan^2\alpha$. 4

(b) Find the equation of the cylinder generated by the line parallel to Z-axis and passing through the curve of intersection of the plane $3x + 2y - z = 4$ and the surface $5x^2 - 2y^2 + 3z^2 = 1$. 4

45. *(a)* A variable plane is parallel to the given plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and meets the axes in A, B and C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{a}{c} + \frac{c}{a}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$ 4

(b) Find the equation of the right circular cylinder whose axis is the line

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{6}$$

and radius is 5 units. 4

OPTION—B

Course No. : MATDSE–502T (B)

(Probability and Statistics)

SECTION—A

Answer any *twenty* of the following questions : $1 \times 20 = 20$

1. Define sample space.

2. Write the sample space of the experiment of throwing a die and tossing a coin simultaneously.

3. Two dice are thrown simultaneously. What is the probability that the numbers that appear are both prime?

4. Define random variable.

5. If $E(X) = 7, E(Y) = 8$, what is $E(X+Y)$?

6. What are Bernoulli trials?

7. Write the p.m.f. of binomial distribution.

8. What is the mean of a Poisson distribution?

9. What is the p.d.f. of exponential distribution?

10. If $X \sim N(1, 2)$, what is σ?

11. What is joint probability mass function?

12. What is marginal probability function?

13. What are independent random variables?

14. What is the expectation of a linear combination of random variables?

15. Define conditional expectation.

16. Define Karl Pearson coefficient of correlation between two random variables.

17. What is regression analysis?

18. What is line of regression?

19. How is covariance between two random variables defined?

20. What is the regression line of Y on X?

21. State Chebyshev's inequality.

22. State weak law of large numbers.

23. State central limit theorem.

24. State Markov's inequality.

25. Define convergence in probability of a sequence of random variables.

SECTION—B

Answer any *five* of the following questions : 2×5=10

26. A bag contains 3 black, 2 yellow and 4 green balls. Two balls are drawn at random. What is the probability that none of them is yellow?

27. Two cards are drawn from a pack of 52 cards. What is the probability that both of them are queens?

28. Compute the mean of a binomial distribution.

29. Write some uses of normal distribution.

30. The joint probability density function of (X, Y) is given by

$$f(x, y) = \begin{cases} \frac{8}{9}xy & , 1 \le x \le y \le 2 \\ 0 & , \text{elsewhere} \end{cases}$$

Find the marginal density function of X.

31. If X is a continuous random variable with $X \ge 0$, then show that $E(X) \ge 0$.

32. Show that the correlation coefficient lies between -1 and 1.

33. If $X, Y \sim \text{BVN}(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ and $\rho = 0$, then show that X and Y are independent.

34. A symmetric die is thrown 600 times. Find a lower bound for the probability of getting 80 to 120 sixes.

35. State Chebyshev's theorem and prove it using Chebyshev's inequality.

SECTION—C

Answer any *five* of the following questions : 8×5=40

36. *(a)* If X and Y are random variables, then show that $E(X+Y) = E(X) + E(Y)$ and $E(XY) = E(X)E(Y)$. 5

(b) Let X be the number of heads in tossing three unbiased coins. Write the probability distribution of X. 3

37. *(a)* A random variable X has the following probability mass function :

X	0	1	2	3	4	5	6
$P(X)$	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$

(i) Find the value of K.

(ii) Find $P(3 < X \leq 5)$.

(iii) Obtain the distribution function of X.

(iv) What is the smallest value of x for which $P(X \leq x) > 0 \cdot 5$? 1+1+1+1=4

(b) Let X be a random variable having p.d.f. given by

$$f(x) = \begin{cases} cx & , \quad 0 \leq c \leq 1 \\ c & , \quad 1 < x < 2 \\ -cx + 3c & , \quad 2 \leq x \leq 3 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find c and the distribution function of X. 4

38. *(a)* Derive the probability function of Poisson distribution. 5

(b) Show that for a negative binomial distribution, the mean is less than variance. 3

39. *(a)* Show that mean = median for a normal distribution. 4

(b) Derive the moment-generating function of an exponential distribution and hence find the mean and variance. 4

40. *(a)* Show that the joint probability function of a two-dimensional random variable is monotonic non-decreasing. 4

(b) If x and y are random variables, then show that $E(X+Y) = E(X) + E(Y)$ provided all the expectations exist. 4

41. *(a)* Let x and y be jointly distributed with density function

$$f(x, y) = \begin{cases} 6(1-x-y) & , \quad x > 0, y > 0, x + y < 1 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

Find the marginal distributions of x and y, hence examine if x and y are independent. 5

(b) Show that the expected value of X is equal to the expectation of the conditional expectation of X given Y, i.e.,

$$E(X) = E(E(X|Y))$$ 3

42. *(a)* Show that the correlation coefficient is independent of change of origin and scale. 4

(b) Derive the equation of line of regression of Y on X. 4

43. Derive the expression for moment-generating function of a bivariate normal distribution. 8

44. *(a)* Prove Chebyshev's inequality. 4

(b) State and prove Lindeberg-Levy theorem. 4

45. *(a)* Prove the weak law of large numbers. 4

(b) State and prove Bernoulli's law of large numbers. 4

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MATHEMATICS

(5th Semester)

Course No. : MATDSE–501T

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

Honours students should answer *either* from Option—A *or* Option—B

Pass students should answer *either* from Option—C or Option—D

OPTION—A

(For Honours students only)

Course No. : MATDSE–501T (A)

(Mechanics)

SECTION—A

Answer any *twenty* of the following as directed : 1×20=20

1. Any system of coplanar forces acting on a rigid body can be reduced ultimately to a single force or a single couple, unless it is in ____.

(Fill in the blank)

2. If three coplanar forces acting on a rigid body be in equilibrium, they must either all three ____ at a point or else all must be ____ to one another.

(Fill in the blanks)

3. Define coefficient of friction.

4. What do you mean by angle of friction?

5. Define cone of friction.

6. Define angular velocity.

7. Write the expressions for radial and cross-radial (transverse) components of acceleration.

8. Define SHM.

9. Where do you find minimum and maximum velocities of a particle executing SHM?

10. Time period of SHM is independent of the amplitude.

(Write True or False)

11. Write the equation of motion when the law of attraction is $\frac{\mu}{x^{5/3}}$.

12. Write the expression of velocity for a particle moving in an elliptical orbit.

13. What do you mean by areal velocity?

14. If velocity of a particle is given by

$$v^2 = \frac{2\mu}{r}$$

is the orbit hyperbolic? If not, what can you say about the orbit?

15. If a particle of mass m falls from rest under gravity (supposed constant) in a medium, whose resistance varies as the square of the velocity, write the equation of motion.

16. Define kinetic energy.

17. Define potential energy.

18. State the principle of work.

19. What is Newton's experimental law of direct impact?

20. If the coefficient of elasticity e is zero, the body is said to be ____ and if $e = 1$, the body is said to be ____.

(Fill in the blanks)

21. Define moment of inertia of a particle about a line.

22. State perpendicular axes theorem.

23. State parallel axes theorem.

24. If the moments and product of inertia about three perpendicular axes are A, B, C and D, E, F, what is the moment of inertia about a line through their meeting point?

25. State D'Alembert's principle.

SECTION—B

Answer any *five* of the following questions : 2×5=10

26. Write any two laws of limiting friction.

27. State a set of necessary and sufficient conditions for the equilibrium of a system of coplanar forces acting on a rigid body.

28. The velocities of a particle along and perpendicular to the radius vector are respectively λ_r and μ_θ from a fixed origin. Find the path.

29. If the tangential and normal accelerations of a particle describing a plane curve be constant throughout, prove that the radius of curvature at any point is given by $\rho = (at+b)^2$.

30. State Kepler's laws of planetary motion.

31. If a particle is projected upwards against gravity (supposed constant) in a resisting medium whose resistance varies as the velocity, obtain the expression for terminal velocity in the medium.

32. State the principle of conservation of momentum.

33. Prove that, impulse = change in momentum.

34. Show that the moment of inertia of a body about a line can be expressed in the form

$$I = \sum m(\vec{r} \times \hat{a})^2$$

where $\hat{a}$ is the unit vector in the direction of the line, the origin being taken on the line.

35. What is the MI of a circular ring of radius a and mass M *(a)* about a diameter and *(b)* about an axis through its centre and perpendicular to its plane?

SECTION—C

Answer any *five* of the following questions : 8×5=40

36. *(a)* A uniform beam of length $2a$ rests in equilibrium against a smooth vertical wall and upon a peg at a distance b from the wall. Show that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$$ 4

(b) Forces P, Q, R and S act along the sides $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$ of a cyclic quadrilateral $ABCD$, taken in order, where A and B are the extremities of a diameter. If they are in equilibrium, then prove that

$$R^2 = P^2 + Q^2 + S^2 + \frac{2PQS}{R}$$ 4

37. *(a)* A uniform ladder rests in limiting equilibrium with its lower end on a rough horizontal plane and its upper end against a smooth vertical wall. If θ be the inclination of the ladder to the vertical, then prove that $\tan\theta = 2\mu$, where μ is the coefficient of friction. 4

(b) A heavy uniform rod rests in equilibrium (limiting) within a fixed rough hollow sphere. If λ be the angle of friction and 2α the angle subtended by the rod at the centre of sphere, show that the inclination θ of the rod to the horizon is given by

$$2\tan\theta = \tan(\alpha+\lambda) - \tan(\alpha-\lambda)$$ 4

38. *(a)* The acceleration of a point moving in a plane curve is resolved into two components, one parallel to the initial line and the other along the radius vector; prove that these components are

$$-\frac{1}{r\sin\theta}\frac{d}{dt}(r^2\dot{\theta})$$

and $$\frac{\cot\theta}{r}\frac{d}{dt}(r^2\dot{\theta}) + \ddot{r} - r\dot{\theta}^2$$ 4

(b) A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'; the particle is slightly displaced towards one of them, show that the time of a small oscillation is

$$\frac{2\pi}{\sqrt{\mu+\mu'}}$$ 4

39. *(a)* Obtain the expressions for tangential and normal components of acceleration of a particle moving in a plane curve. 5

(b) A body performing SHM in a straight line OAB has velocity zero when at points A and B whose distances from O are a and b respectively and has velocity v when half-way between them. Show that the complete period is

$$\frac{\pi(b-a)}{v}$$ 3

40. *(a)* A particle moves in a straight line from a distance a towards the centre of force, the force being $\frac{\mu}{x^3}$ at a distance x from the centre. Show that it reaches the centre in time $\frac{a^2}{\sqrt{\mu}}$. 5

(b) If v_1 and v_2 are the linear velocities of a planet when it is respectively nearest and farthest from the sun, prove that

$$(1-e)v_1 = (1+e)v_2$$ 3

41. *(a)* A planet is describing an ellipse about the sun as focus, show that its velocity away from the sun is greatest when the

radius vector to the planet is at right angles to the major axis of the path and that it then is

$$\frac{2\pi aeT}{\sqrt{1-e^2}}$$

where $2a$ is the major axis, e the eccentricity and T the periodic time. 4

(b) A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is

$$\frac{V^2}{g}[\lambda - \log(1+\lambda)]$$

where V is the terminal velocity of the particle and λV is the initial vertical velocity. 4

42. *(a)* A shell lying in a straight smooth horizontal tube suddenly explodes and breaks into portions of masses m and m'. If d is the distance apart of the masses after a time t, show that the work done by the explosion is

$$\frac{1}{2}\cdot\frac{mm'}{m+m'}\cdot\frac{d^2}{t^2}$$ 5

(b) If two equal and perfectly elastic spheres moving with velocities u and u' impinge directly, show that they interchange their velocities after impact. 3

43. (a) A sphere impinges directly on an equal sphere at rest. Show that a portion

$$\frac{1}{2}(1-e^2)$$

of the original KE is lost in the impact. 4

(b) A particle falls from a height h in time t upon a fixed horizontal plane. Show that the whole distance (up and down) described by the particle before it has finished rebound is

$$\frac{(1+e^2)}{(1-e^2)}h$$

4

44. Show that the MI of an elliptic area of mass M and semi-axes a and b about a tangent is $\frac{5}{4}Mp^2$, where p is the perpendicular from the centre on the tangent. 8

45. A uniform rod OA, of length $2a$, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant α to OZ, show that the value of α is either zero or

$$\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$$

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OPTION—B

(For Honours students only)

Course No. : MATDSE–501T (B)

(Number Theory)

SECTION—A

Answer any *twenty* of the following questions : 1×20=20

1. Write the general solution of the linear diophantine equation $ax+by=c$, when $\gcd(a, b)|c$.

2. How many positive solutions does the equation $2x+10y=17$ have?

3. State Goldbach conjecture.

4. If $ca \equiv cb \pmod{n}$, is it always true that $a \equiv b \pmod{n}$?

5. Find the unit's digit in 3^{100}.

6. Calculate $\tau(180)$, where $\tau(n)$ denotes the number of positive divisors of n.

7. State Möbius inversion formula.

8. Define $\tau(n)$ and $\sigma(n)$ in terms of its divisors.

9. Write the value of $\tau(n)$ in terms of its prime factors.

10. Find $\sigma(360)$.

11. Define Euler's phi-function $\phi(n)$.

12. Compute $\phi(16)$.

13. Can $3^{40} \equiv 2 \pmod{100}$?

14. What is the value of $\sum_{d|n} \phi(d)$?

15. Write the formulae of $\phi(n)$ in terms of the prime factors of n.

16. Find the order of 2 modulo 7.

17. Find the number of primitive roots of 9.

18. Does 8 have any primitive root?

19. Does 15 have any primitive root?

20. Define primitive root of an integer.

21. Define perfect number.

22. Give example of an amicable pair of numbers.

23. Is every Fermat number of the form $2^{2^n}+1$ prime for $n \geq 1$?

24. State Pepin's test for primality of Fermat numbers.

25. Write the general solution of $x^2 + y^2 = z^2$.

SECTION—B

Answer any *five* of the following questions : 2×5=10

26. State Fermat's theorem and Wilson's theorem.

27. Find the remainder when 41^{65} is divided by 7.

28. Show that τ is a multiplicative function.

29. Show that $\tau(n) \leq 2\sqrt{n}$ for any $n \geq 1$.

30. For $n \geq 2$, show that $\phi(n)$ is even integer.

31. Show that $\phi(p^k) = p^k - p^{k-1}$ for any prime p.

32. List all numbers that have primitive root.

33. Show that 2 is not a primitive root of $2^{2^n}+1$ for any $n \geq 2$.

34. Show that if p and $2p+1$ are primes, then $2p+1 | M_p$ or $2p+1 | M_p + 2$ but not both, where $M_p = 2^p - 1$ is the pth Mersenne number.

35. If $a^k - 1$ is prime for $a > 0$ and $k \geq 2$, then show that $a = 2$ and k is a prime.

SECTION—C

Answer any *five* of the following questions : 8×5=40

36. *(a)* Show that if $\gcd(a, b) = 1$, the linear diophantine equation has infinitely many positive solutions. 4

(b) Prove that $53^{103} + 103^{53}$ is divisible by 39. 4

37. *(a)* Let p be a prime and a be any integer. Show that $a^p \equiv a \pmod p$. 4

(b) Solve the congruence $17x \equiv 9 \pmod{276}$ by Chinese remainder theorem. 4

38. *(a)* If f is a multiplicative function, then show that

$$F(n) = \sum_{d|n} f(d)$$

is also multiplicative. 4

(b) Show that

$$n^{\frac{1}{2}\tau(n)} = \prod_{d|n} d$$ 4

39. *(a)* Derive a formula for $\sigma(n)$ in terms of the prime factors of n. 4

(b) Show that

$$\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$$ 4

40. *(a)* If $\gcd(a, n) = 1$, then show that

$$a^{\phi(n)} \equiv 1 \pmod n$$ 4

(b) Show that for $n > 1$, the sum of positive integers less than n and relatively prime to n is $\frac{1}{2} n\phi(n)$. 4

41. *(a)* Show that

$$\phi(n) = \left(\sum_{d|n} \frac{\mu(d)}{d} \right) \times n$$ 4

(b) Show that ϕ is a multiplicative function. 4

42. *(a)* If p is a prime number and $d|p-1$, then show that the congruence $x^d - 1 \equiv 0 (\text{mod } p)$ has exactly d solutions. 4

(b) If $m > 2$ and $n > 2$, then show that mn has no primitive roots, whenever $\gcd(m, n) = 1$. 4

43. *(a)* If $d|p-1$ and p is prime, show that there are exactly $\phi(d)$ incongruent integers having order d modulo p. 4

(b) For $k \geq 3$, show that 2^k has no primitive roots. 4

44. *(a)* Show that every even perfect number has the form $2^{k-1}(2^k - 1)$ where $k > 1$ and $2^k - 1$ is prime. 4

(b) Show that every even perfect number ends with 6 or 8. 4

45. *(a)* If p is an odd prime, then show that any prime divisor of M_p is of the form $2kp+1$ for $k \geq 1$. 4

(b) Show that an odd prime p is a sum of two squres iff $p \equiv 1 (\text{mod } 4)$. 4

OPTION—C

(For Pass students only)

Course No. : MATDSE–501T (C)

(Matrices)

SECTION—A

Answer any *twenty* of the following questions : 1×20=20

1. Is the set {(1, 2), (3, 6)} a basis of $\mathbb{R}^2$? Justify.

2. Express (5, 6) as a linear combination of the vectors (1, 1) and (1, 0).

3. Is the set $V = \{(x, y) | x + y = 2\}$ a subspace of $\mathbb{R}^2$? Justify.

4. What is the dimension of the subspace

$$W = \{(x, y, z) | x + y = 0\}$$

of $\mathbb{R}^3$?

5. When is a set of vectors in $\mathbb{R}^3$ said to be linearly independent?

6. What is the reflection of the point (3, 2) about X-axis?

7. If the vector (1, 2) is rotated anticlockwise by 30°, what is the transformed vector?

8. What is the linear transformation that represents reflection about Y-axis?

9. Give one eigenvalue of a singular matrix.

10. What is the linear transformation that represents reflection about the line $y = x$?

11. Define skew-Hermitian matrix.

12. Give example of an orthogonal matrix.

13. Define echelon form of a matrix.

14. What are the three elementary transformations of a matrix?

15. Define idempotent matrix.

16. Define inverse of a matrix.

17. When is a matrix said to be diagonalizable?

18. Show that every diagonal matrix is diagonalizable.

19. Is the matrix

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 7 & 8 \\ 0 & 0 & 5 \end{pmatrix}$$

diagonalizable? Justify.

20. Let A be a matrix and P is a non-singular matrix such that $PAP^{-1} = \text{diag}(3, 4, -1)$. What is the determinant of A?

21. What is the rank of an identity matrix of order 10?

22. A is an $n \times n$ singular matrix. Can it have rank n? Justify your answer.

23. A is a 5×7 matrix. Can it have 6 linearly independent columns? Justify your answer.

24. Write the conditions of consistency of a non-homogeneous system of linear equations.

25. Can a system of homogeneous linear equations be inconsistent? Justify your answer.

SECTION—B

Answer any *five* of the following questions : 2×5=10

26. Show that the vectors (1, 2, 3), (0, 1, 4), (1, 2, 4) are linearly independent.

27. Show that {(1, 1), (0, 1)} is a basis of $\mathbb{R}^2$.

28. Find the matrix of transformation representing reflection about X-axis.

29. Define eigenvalue and eigenvector of a matrix.

30. Reduce the matrix A to echelon form :

$$A = \begin{pmatrix} 5 & 7 & 8 & 9 \\ 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

31. If x and y are solutions of a linear homogeneous system of equations $AX = 0$, then show that any linear combination of x and y is also a solution of the same system.

32. How can diagonalizability help in computing higher powers of a matrix?

33. Explain the process of obtaining matrix inverse using elementary row operations.

34. Check the consistency of the system

$$2x + y + 3z = 1$$
$$3x - 2y + z = 7$$
$$5x - y + 4z = 9$$

35. For what values of k does the following system of equations have non-trivial solution?

$$x + y + z = 0$$
$$2x + 2y + kz = 0$$
$$x + 3y + 3z = 0$$

SECTION—C

Answer any *five* of the following questions : 8×5=40

36. *(a)* If V and W are subspaces of $\mathbb{R}^3$, then show that $V \cap W$ is also a subspace of $\mathbb{R}^3$. Is the same true for union? Justify. 3+1=4

(b) Show that $V = \{(x, y) \mid 2x + 3y = 0\}$ is a subspace of $\mathbb{R}^2$. 4

37. *(a)* Express (2, 3, 4) as a linear combination of (1, 3, 2), (1, 1, 0) and (1, 2, −2). 5

(b) Show that a set containing a single non-zero vector is linearly independent, whereas {0} is linearly dependent. 3

38. *(a)* Show that 0 is an eigenvalue of a matrix A if and only if A is singular. 3

(b) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

5

39. *(a)* Show that eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent. 5

(b) Find the characteristic polynomial of

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

Express it as a polynomial in decreasing powers of the indeterminate (i.e., variable say λ). 3

40. *(a)* Define nilpotent matrix. Provide an example. 2

(b) Define idempotent matrix. Provide an example. 2

(c) Solve the homogeneous system of equations : 4

$$\begin{aligned} 2x+3y-z&=0 \\ x-y-z+w&=0 \\ 3x+y-2z+3w&=0 \\ 3x+2y-3z+3w&=0 \end{aligned}$$

41. *(a)* Reduce the matrix to normal form : 4

$$\begin{pmatrix} 1 & 3 & 4 & 7 \\ 2 & 1 & 3 & 0 \\ 4 & 7 & 11 & 14 \end{pmatrix}$$

(b) Solve the system of equations using Gaussian elimination : 4

$$\begin{aligned} 2x+y+z&=7 \\ 3x-y+3z&=10 \\ -x+3y-2z&=-1 \end{aligned}$$

42. *(a)* Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

using elementary row operations. 4

(b) Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$$

given that two eigenvectors of A are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ 4

43. (a) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

using elementary column operations. 3

(b) Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$ 5

44. A manufacturer produces three types of products A, B and C. Each product of type A requires 12 hours of cutting and 5 hours of finishing. Each product of type B requires 10 hours of cutting and 3 hours of finishing. Each product of type C requires 6 hours of cutting and 1 hour of finishing. On a daily basis, the manufacturer has 440 working hours available for cutting and

120 working hours for finishing. Formulate the problem mathematically using linear equations to determine how many products of each type can be produced so that all the work power is used. Also, check if the resulting system is consistent. Check if it has a unique solution. Write the general solution in case the system is consistent. 3+5=8

45. The measure of the largest angle of a triangle is 30° less than the sum of the other two angles twice the smallest angle is 9° more than the largest angle. Can we determine the measures of all the angles from this information? Formulate the problem as an equivalent system of linear equations. Check if the resulting system is consistent. Check if it has a unique solution. Write the general solution in case the system is consistent. 3+5=8

OPTION—D

(For Pass students only)

Course No. : MATDSE–501T (D)

(Linear Algebra)

SECTION—A

Answer any *twenty* of the following as directed : 1×20=20

1. Define linear dependence of vectors.

2. Let $V(F)$ be a vector space and 0 be the zero vector of V. Then show that $\alpha 0 = 0,\ \forall \alpha \in F$.

3. When is the union of two subspaces of a vector space also a subspace?

4. Prove that every singleton set containing non-zero vector is LI.

5. Define basis of a vector space.

6. Define linear operator.

7. Define kernel of a linear transformation.

8. Define range space of a linear transformation.

9. If V is a vector space and $T : V \to V$ is a linear operator, then what is the intersection of range of T and the null space of T?

10. State Sylvester's law of nullity.

11. Let T_1 and T_2 be two linear transformations from $V(F)$ to $U(F)$. Then $T_1 + T_2$ is also a linear transformation.
(Write True or False)

12. Let $V(R)$ be the vector space of all complex numbers $a + ib$ over the field of reals R and let T be a mapping from $V(R)$ to $V_2(R)$ defined as $T(a+ib) = (a, b)$. Then show that T is onto.

13. Give an example of homomorphism on vector space.

14. Define homomorphism on vector space.

15. Write down the necessary and sufficient condition for a linear transformation T on a vector space $V(F)$ to be invertible.

16. Define characteristic value.

17. Define characteristic vector.

18. Distinct characteristic vectors of T corresponding to distinct characteristic value of T are linearly independent.

(Write True or False)

19. Write down the characteristic equation of the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

20. State Cayley-Hamilton theorem.

21. Define norm of a vector in inner product space.

22. Define unit vector of a vector in inner product space.

23. Define orthogonality.

24. State Cauchy-Schwarz inequality.

25. State Bessel's inequality.

SECTION—B

Answer any *five* of the following questions : 2×5=10

26. Let $V(F)$ be a vector space and 0 be the zero vector of V. Then show that

$\alpha v = 0 \Rightarrow \alpha = 0$ or $v = 0$, $\forall \alpha \in F$ and $\forall v \in V$

27. Show that the vectors $(1, 1, 0, 0)$, $(0, 1, -1, 0)$, $(0, 0, 0, 3)$ in R^4 are linearly independent.

28. Let $T : U \to V$ be a linear transformation. Then prove that

$$T(-u) = -T(u), \ \forall u \in U$$

29. The mapping $T : V_2(R) \to V_3(R)$ defined as

$$T(a, b) = (a+b, a-b, b)$$

is a linear transformation. Find the null space of T.

30. If $T : U \to V$ is a homomorphism, then prove that $T(0) = 0'$ where 0 and $0'$ are the zero vectors of U and V respectively.

31. Define isomorphism of a vector space.

32. Let T be a linear operator on a finite-dimensional vector space V and let λ be the characteristic value of T. Then prove that the set $W_\lambda = \{v \in V \mid Tv = \lambda v\}$ is a subspace of V.

33. If λ is a characteristic value of an invertible transformation T, then show that λ^{-1} is a characteristic value of T^{-1}.

34. In an inner product space $V(F)$, prove that

$$||\alpha x|| = |\alpha| \, ||x||$$

35. Define orthonormal set in inner product space.

SECTION—C

Answer any *five* of the following questions : 8×5=40

36. *(a)* Prove that the necessary and sufficient condition for a non-empty subset W of a vector space $V(F)$ to be a subspace of V is $\alpha x + \beta y \in W$, $\forall \alpha, \beta \in F$ and $\forall x, y \in W$. 5

(b) Show that the vectors $(1, 2, 1)$, $(2, 1, 0)$, $(1, -1, 2)$ form a basis of R^3. 3

37. In any finite-dimensional vector space, prove that any two bases have the same number of elements. 8

38. *(a)* The function $T: V_3(R) \to V_2(R)$ defined by

$$T(a, b, c) = (a, b), \forall a, b, c \in R$$

Then show that T is a linear transformation from $V_3(R)$ into $V_2(R)$. 3

(b) If $U(F)$ and $V(F)$ are two vector spaces and $T: U \to V$ is a linear transformation, then show that range of T is a subspace of V and kernel of T is a subspace of U. 5

39. *(a)* Show that the mapping $T: V_2(R) \to V_3(R)$ defined as $T(a, b) = (a+b, a-b, b)$ is a linear transformation from $V_2(R)$ into $V_3(R)$. 3

(b) Let T be the linear operator on R^2 defined by

$$T(x, y) = (4x - 2y, 2x + y)$$

Compute the matrix of T relative to the basis $\{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1, 1)$ and $\alpha_2 = (-1, 0)$. 5

40. Prove that two finite-dimensional vector spaces over the same field are isomorphic if and only if they are of the same dimension. 8

41. If $T: U \to V$ is an isomorphism of the vector space U into V, then prove that a set of vectors $\{T(u_1), T(u_2), \cdots, T(u_r)\}$ is linearly independent if and only if the set $\{u_1, u_2, \cdots, u_r\}$ is linearly independent. Give example to show that the same does not hold if T is not isomorphism. 6+2=8

42. Let T be a linear operator on a finite-dimensional vector space V. Then prove that the following are equivalent : 8

(a) λ is a characteristic value of T

(b) The operator $T - \lambda I$ is singular

(c) $|T - \lambda I| = 0$

43. Find all eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

8

44. *(a)* If x, y are vectors in an inner product space V, then prove that

$$||x + y|| \leq ||x|| + ||y||$$

5

(b) If in an inner product space the vectors x and y are linearly dependent, then show that $|(x, y)| = ||x|| \, ||y||$. 3

45. Prove that every finite-dimensional inner product space has an orthonormal basis. 8

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