CENTRAL LIBRARY N.C.COLLEGE

**2021/TDC/CBCS/ODD/
MATHCC–501T/329**

**TDC (CBCS) Odd Semester Exam., 2021
held in March, 2022**

MATHEMATICS

(5th Semester)

Course No. : MATHCC–501T

(Topology)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Define bounded metric space and give an example.

2. Show that in a discrete metric space every set is open.

3. Let X be a metric space and let G be an open set in X. Prove that $G \cap A = \phi$ if and only if $G \cap \overline{A} = \phi$.

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4. Show that every convergent sequence in a metric space is a Cauchy sequence.

5. Give an example of a metric space which is not complete.

6. Define continuity of a function in metric space.

7. Define a topological space and give an example.

8. Define discrete and indiscrete topologies.

9. Write two distinct topologies on $X = \{a, b, c\}$.

10. Define metrizable space.

11. Show with an example that the union of two topologies on a set may not be a topology.

12. Define interior and exterior points of a set in a topological space.

13. Find the condition that a function f from topological spaces X to a topological space Y is not continuous at a point $x \in X$.

14. Define homeomorphism of a function in topological space.

15. Show that identity function is continuous.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. (a) Let X be a non-empty set. Show that the function $d: X \times X \to \mathbb{R}$ is a metric on X if and only if d satisfies the conditions—

(i) $d(x, y) = 0 \Leftrightarrow x = y \ \forall x, y \in X$;

(ii) $d(x, y) \le d(x, z) + d(y, z) \ \forall \ x, y, z \in X$. 5

(b) If (X, d) be a metric space and A is a subset of X, then show that $\overline{A}$ is the smallest closed set containing A. 5

17. (a) Let $S(x_0, r)$ be an open sphere in a metric space (X, d). Prove that to each $p \in S(x_0, r)$ there exists $r' > 0$ such that $S(p, r') \subseteq S(x_0, r)$. 5

(b) Let A be a subset of a metric space (X, d). Prove that $\overline{A} = A \cup D(A)$. 5

18. (a) If x and y are two points of a metric space (X, d) such that the sequences $< x_n >$ and $< y_n >$ in (X, d) converges to x and y respectively, then prove that the sequence $< d(x_n, y_n) >$ converges to $d(x, y)$. 5

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(b) Let (X, d) and (Y, ρ) be metric spaces. Show that a function $f: X \to Y$ is continuous if and only if for every subset $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$. 5

19. (a) Show that $\mathbb{R}$ (with the usual metric) is a complete metric space. 5

(b) Let (X, d) and (Y, ρ) be metric spaces and $f: X \to Y$ a mapping. If f is continuous at $x \in X$, then show that for every open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(U) \subseteq V$. 5

20. (a) Let $\mathbb{N}$ be the set of all natural numbers and T the family of subsets of $\mathbb{N}$ consisting ϕ and the sets of the form

$$T_n = \{n, n+1, n+2, \cdots\}, n \in \mathbb{N}$$

Prove that T is a topology for $\mathbb{N}$. 5

(b) Let $\mathbb{R}$ be the set of all real numbers and T the collection of all those subsets S of $\mathbb{R}$ such that either $S = \phi$ or $S \neq \phi$, then for each $x \in S$ there exists a right half open interval H such that $x \in H \subseteq S$. Prove that T is a topology for $\mathbb{R}$. 5

21. (a) Show that an arbitrary intersection of closed subsets of a topological space is a closed set. 5

(b) Define upper limit topology on $\mathbb{R}$. Establish that it is a topology. 5

22. (a) Show that the intersection of arbitrary collection of topologies on a set is also a topology. 5

(b) Show that every metric space is a topological space. 5

23. (a) Let (X, T) be a topological space and $A \subseteq X$. Show that int(A) is the largest open subset of X containing A. 5

(b) Let A and B be any two subsets of a topological space. Prove that—

(i) $A \subseteq B \Rightarrow D(A) \subseteq D(B)$;

(ii) $D(A \cup B) = D(A) \cup D(B)$. 2+3=5

24. (a) Let X and Y be topological spaces. Show that a function $f: X \to Y$ is continuous iff the inverse image under f of every open set in Y is open in X. 5

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(b) Let T_1 and T_2 denote the discrete and usual topology respectively on $\mathbb{R}$. Show that the function $f:(\mathbb{R}, T_1) \to (\mathbb{R}, T_2)$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \in \mathbb{R} - Q \end{cases}$$

is continuous. 5

25. (a) Prove that $f : X \to Y$ is a homomorphism if and only if f is both continuous and open. 5

(b) Prove that a constant function from one topological space to another is continuous. 5

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MATHCC–502T/330

TDC (CBCS) Odd Semester Exam., 2021
held in March, 2022

MATHEMATICS

(5th Semester)

Course No. : MATHCC–502T

(Multivariate Calculus)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Check if the limit exists

$$\operatorname*{Lt}_{(x,\ y)\to(0,\ 0)} \frac{x^2 + xy}{2xy - y^2}$$

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2. Check the continuity of f at origin, where

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 1 \quad , & (x, y) = (0, 0) \end{cases}$$

3. Evaluate f_x and f_y if

$$f(x, y) = x^3 y + \sin(xy^2)$$

4. Define extreme value of a function of two variables. Give an example.

5. Find the stationary points of

$$f(x, y) = y^2 + 4xy + 3x^2 + x^3$$

6. Can you mention an extreme value of the following function?

$$f(x, y) = |x| + |y|, \ (x, y) \in \mathbb{R}^2$$

Justify your answer.

7. For the function

$$\vec{F} = yz^2 \hat{i} + xy\hat{j} + yz\hat{k}$$

verify that div curl $\vec{F} = 0$.

8. Evaluate

$$\iint (x + y)\, dx\, dy$$

over the area bounded by the lines $y = x$, $x = 3$ in the first quadrant.

9. Sketch the region of integration for the integral

$$\int_0^{\pi} \int_0^{\sin x} y\, dy\, dx$$

10. Compute the Jacobian of transformation from Cartesian to spherical polar coordinates.

11. Change the order of the integration

$$\int_0^1 dx \int_x^{\sqrt{x}} f(x, y)\, dy$$

12. Evaluate the line integral

$$\int_C xy\, dx$$

where C is the arc of the parabola $x = y^2$ from $(1, -1)$ to $(1, 1)$.

13. State Green's theorem in $\mathbb{R}^2$.

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14. Using Green's theorem, deduce the expression for area of a domain bounded by a contour C regular with respect to both the axes.

15. State Gauss' divergence theorem.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. (a) Show that if f and g are two functions defined on some neighbourhood of (a, b) such that

$$\underset{(x,\ y)\to(a,\ b)}{\text{Lt}} f(x, y) = l \text{ and } \underset{(x,\ y)\to(a,\ b)}{\text{Lt}} g(x, y) = m$$

then

$$\underset{(x,\ y)\to(a,\ b)}{\text{Lt}} (f+g)(x, y) = l + m$$ 5

(b) Show that the function f is continuous, where

$$f(x, y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$ 5

17. (a) Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

possesses both the partial derivatives at $(0, 0)$ but is not differentiable at $(0, 0)$. 5

(b) Define directional derivative of a function of two variables at a point (a, b) in the direction of unit vector $\hat{u}$. Derive the directional derivative of

$$f(x, y) = x^2 + y^2$$

at (a, b) in the direction of unit vector $\hat{u} = u_1\hat{i} + u_2\hat{j}$. 2+3=5

18. (a) Investigate the function

$$f(x, y) = 2x^4 - 3x^2y + y^2$$

for extreme values. 5

(b) Find the shortest distance from the origin to the hyperbola

$$x^2 + 8xy + 7y^2 = 225, \ z = 0$$ 5

19. (a) If $2x + 3y + 4z = a$, show that the maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$. 5

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(b) Find an extreme value of the function

$$f(x, y) = x^2 + 3xy + y^2 + x^3 + y^3$$ 5

20. (a) Evaluate

$$\iint_R \frac{dx\,dy}{(x+y+1)^2}$$

over the rectangle $R = [0, 1; 0, 1]$. 5

(b) Evaluate

$$\iint_R (x^2 + y^2)\,dx\,dy$$

over the region R bounded by the parabolas $y = x^2$ and $y^2 = x$. 5

21. (a) Evaluate

$$\iint_C x^3 y^2\,dx\,dy$$

where C is the circular disc $x^2 + y^2 \le a^2$. 5

(b) Using double integration, show that the area of a circle of radius r is πr^2. 5

22. (a) Compute the integral

$$\iiint_E xyz\,dx\,dy\,dz$$

where E is bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$. 5

(b) Compute the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ 5

23. (a) Find the line integral

$$\int_C (x-y)^3\,dx + (x-y)^3\,dy$$

where C is the circle $x^2 + y^2 = a^2$ in counterclockwise direction. 5

(b) Evaluate

$$\iint_R f(x, y)\,dy\,dx$$

over the rectangle $R = [0, 1; 0, 1]$, where

$$f(x, y) = \begin{cases} x + y, & x^2 < y < 2x^2 \\ 0, & \text{elsewhere} \end{cases}$$ 5

24. (a) Compute the line integral

$$\int_C (1 - x^2)y\,dx + (1 + y^2)x\,dy$$

where C is $x^2 + y^2 = a^2$, using Green's theorem. 5

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(b) Evaluate the surface integral

$$\iint_S x\,dy\,dz + dz\,dx + xz^2\,dx\,dy$$

where S is the outer part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant. 5

25. (a) Apply Stokes' theorem to evaluate

$$\int_C y\,dx + z\,dy + x\,dz$$

where C is the curve

$$x^2 + y^2 + z^2 - 2ax - 2ay = 0,\ x + y = 2a$$ 5

(b) Use Gauss' divergent theorem to evaluate

$$\iint_S y^2 z\,dx\,dy + xz\,dy\,dz + x^2 y\,dz\,dx$$

where S is the outer side of the surface in first octant formed by the paraboloid of revolution $z = x^2 + y^2$, cylinder $x^2 + y^2 = 1$ and the coordinate planes. 5

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