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2021/TDC/CBCS/ODD/ MATHCC–301T/325

TDC (CBCS) Odd Semester Exam., 2021 held in March, 2022

MATHEMATICS

(3rd Semester)

Course No. : MATHCC–301T

(Theory of Real Functions)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Show that $\lim_{x\to 0} \frac{1}{x}$, $x>0$ does not exist.

2. Using squeeze theorem, show that

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right)=0$$

3. Show that $\lim_{x\to 0} \frac{1}{x^2} = \infty$

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4. State the sequential criterion for continuity.

5. Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}1, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

is not continuous at any point of $\mathbb{R}$.

6. Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=|x|$, $x\in\mathbb{R}$ is continuous at every point of $\mathbb{R}$.

7. Let $I\subseteq\mathbb{R}$ and $f:I\to\mathbb{R}$ has a derivative at $c\in I$. Then prove that f is continuous at c.

8. Show that the function

$$f(x)=\begin{cases}x^2\sin\frac{1}{x}, & \text{for } x\neq 0\\ 0, & \text{for } x=0\end{cases}$$

is differentiable at all $x\in\mathbb{R}$.

9. Show that $-x\le\sin x\le x$, for all $x\ge 0$.

10. Define uniformly continuous function and give an example.

11. Show that if a function f is uniformly continuous on $A\subseteq R$, then it is continuous on A.

12. Define Lipschitz function and give an example.

13. Show that $\dfrac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}=\cot\beta$, where $0<\alpha<\theta<\beta<\dfrac{\pi}{2}$ by using Cauchy's mean value theorem.

14. Define a convex function.

15. Expand $\cos x$ in ascending powers of x.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. *(a)* Let $A\subseteq R$, $c\in\mathbb{R}$ is a cluster point of A and $f:A\to R$ be a function. Define limit of f at c. Further show that if $f:A\to R$ and c is a cluster point of A, then f can have at the most one limit at c. 1+4=5

(b) If $A\subseteq R$ and $f:A\to R$ has limit at $c\in\mathbb{R}$, then show that f is bounded on some neighbourhood of c. 5

17 *(a)* Let $A\subseteq R$, f and g be functions on A to $\mathbb{R}$ and $c\in\mathbb{R}$ be a cluster point of A. If $\lim_{x\to c} f=L$ and $\lim_{x\to c} g=M$, then prove that

$$\lim_{x\to c}(f+g)=L+M$$ 5

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(b) State squeeze theorem on limits. Using squeeze theorem, show that

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1$$ 2+3=5

18. (a) Let $A\subseteq R$, let $f:A\to\mathbb{R}$ and let $|f|$ be defined by $|f|(x):=|f(x)|$ for $x\in A$. Prove that if f is continuous on A, then $|f|$ is also continuous on A. Show with an example that if $|f|$ is continuous, then f may not be continuous. 5

(b) Let $A, B\subseteq\mathbb{R}$, let $f:A\to\mathbb{R}$ be continuous on A, and let $g:B\to R$ be continuous on B. If $f(A)\subseteq B$, then prove that the composite function $g\circ f:A\to R$ is continuous on A. 5

19. (a) Let $I:=[a, b]$ be a closed bounded interval and let $f:I\to\mathbb{R}$ be continuous on I. Prove that f is bounded on I. 5

(b) Let I be an interval and let $f:I\to\mathbb{R}$ be continuous on I. If $a, b\in I$ and if $k\in\mathbb{R}$ satisfies $f(a)<k<f(b)$, then prove that there exists a point $c\in I$ between a and b such that $f(c)=k$. 5

20. (a) Let c be an interior point of an interval I at which $f:I\to R$ has a relative extremum. If the derivative of f at c exists, then prove that $f'(c)=0$. 5

(b) Let f be a continuous function on the closed interval $I=[a, b]$ and that f is differentiable on the open interval (a, b) and that $f'(x)=0$ for $x\in(a, b)$. Then prove that f is constant on I. 5

21. (a) State and prove Darboux's theorem. 1+4=5

(b) Let $I\subseteq\mathbb{R}$ be an interval, let $c\in I$ and let $f:I\to R$ and $g:I\to R$ be two functions that are differentiable at c. Then prove that the function fg is differentiable at c and $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$. 5

22. (a) Prove that if $f:A\to R$ is a Lipschitz function, then f is uniformly continuous on A. Give an example to show that a uniformly continuous function may not be a Lipschitz function with justification. 3+2=5

(b) Prove that if $f:A\to R$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$. 5

23. *(a)* Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then prove that f is uniformly continuous on I. 5

(b) Prove that a function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the end points a and b such that the extended function is continuous on $[a, b]$. 5

24. *(a)* State and prove Cauchy's mean value theorem. 5

(b) Show that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots \text{ for } |x| < 1$$ 5

25. *(a)* State and prove Taylor's theorem with the Lagrange form of the remainder. 5

(b) Show that $1 - \frac{1}{2}x^2 \leq \cos x$ for all $x \in \mathbb{R}$ using Taylor's theorem. 5

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2021/TDC/CBCS/ODD/
MATHCC–302T/326

TDC (CBCS) Odd Semester Exam., 2021
held in March, 2022

MATHEMATICS

(3rd Semester)

Course No. : MATHCC–302T

(Group Theory)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Define quaternion group.

2. If G is a group, then show that

$$(xy)^{-1} = y^{-1}x^{-1} \; \forall \, x, y \in G$$

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3. Show that identity element in a group is unique.

4. Define centre of a group.

5. Prove that a non-empty subset H of a group G is a subgroup of G iff $HH^{-1} = H$.

6. What do you mean by product of two subgroups?

7. Find the generators of the group $G = \{1, -1, i, -i\}$ under the operation multiplication.

8. Is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

a transposition?

9. When is an additive group G said to be cyclic?

10. Define factor group.

11. State Fermat's little theorem.

12. If H is a subgroup of a group G, then show that $aH = bH$ iff $a^{-1}b \in H$.

13. What do you mean by group homomorphism?

14. If $f : G \to G'$ is an isomorphism, what is the Kernel of f?

15. Let $f : G \to G'$ be a homomorphism. Show that $f(G)$ is a subgroup of G'.

SECTION—B

Answer any *five* of the following questions : 10×5=50

16. *(a)* Show that

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R, ad - bc \neq 0 \right\}$$

is a group with respect to matrix multiplication. Is this group abelian? 5

(b) Prove that if G is an abelian group, then for all $a, b \in G$ and all integers n, $(ab)^n = a^n b^n$. 5

17. *(a)* *(i)* Define order of a group.

(ii) If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for $a, b \in G$, find $O(b)$. 2+4=6

(b) Let G be a group and $a, b \in G$. Then show that the equation $ax = b$ has unique solution in G. 4

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18. (a) Let H and K be any two subgroups of a group G. Then show that HK is a subgroup of G iff $HK = KH$. 5

(b) Define order of an element of a subgroup. Show that a non-empty subset H of a finite group G is a subgroup of G iff $HH = H$. 2+3=5

19. (a) What do you mean by normalizer of a group? Show that the intersection of arbitrary collection of subgroups of a group is a subgroup of the group. 2+3=5

(b) Prove that the set

$$H = \{x \in G \mid gx = xg,\ g \in G\}$$

is a subgroup of a group G. Also prove that G is abelian $\Leftrightarrow G = H$. 2½+2½=5

20. (a) Prove that a group of order n is cyclic iff it has an element of order n. 5

(b) What do you mean by composition of two permutations? If $f = (1\ 3\ 5)$ and $g = (2\ 6\ 7)$ be two disjoint cycles on a set S having 7 elements, check whether $f \circ g = g \circ f$ or not. 2+3=5

21. (a) Define alternating group, even permutation and odd permutation. 2+2+2=6

(b) When is a group said to be abelian? Show that every cyclic group is abelian. 1+3=4

22. (a) What do you mean by right coset of a subgroup of a group? Prove that any two right cosets are either disjoint or identical. 2+3=5

(b) Define normal subgroup of a group. Prove that a subgroup H of a group G is normal iff $gHg^{-1} = H\ \forall g \in G$. 1+4=5

23. (a) What do you mean by index of a subgroup? Show that if H is a subgroup of a group G, such that $i_G(H) = 2$, then H is normal in G. 2+3=5

(b) State and prove Lagrange's theorem. 1+4=5

24. (a) State and prove Cayley's theorem. 5

(b) Show that if G be a cyclic group, then the automorphism of G is abelian. 5

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25. *(a)* State and prove fundamental theorem of homomorphism. 6

(b) Show that union of two normal subgroups may not be a normal subgroup. 4

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MATHCC–303T/327

TDC (CBCS) Odd Semester Exam., 2021
held in March, 2022

MATHEMATICS

(3rd Semester)

Course No. : MATHCC–303T

(PDE and System of ODE)

Full Marks : 50
Pass Marks : 20

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* questions : 2×10=20

1. Form the partial differential equation of all spheres whose centre lies on the X-axis.

2. Solve :

$$\frac{\partial^2 z}{\partial x^2} = 0$$

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3. Form the partial differential equation by eliminating arbitrary function from the equation

$$z = f(x^2 + y^2)$$

4. Solve $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$.

5. Solve $\sqrt{x}\frac{\partial z}{\partial x} + \sqrt{y}\frac{\partial z}{\partial y} = \sqrt{z}$.

6. Solve $a\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = z$.

7. Describe the classification of second-order linear partial differential equation.

8. Classify the following PDE :

$$(1+x)^2 \frac{\partial^2 u}{\partial x^2} - 4x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = x$$

9. Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = 0$.

10. What is boundary value problem?

11. Write down two assumptions for deriving one-dimensional wave equation.

12. Write down one-dimensional wave equation.

13. Show that the ordered pair of functions defined for all t by $(e^{5t}, -3e^{5t})$ is a solution of the system

$$\frac{dx}{dt} = 2x - y \text{ and } \frac{dy}{dt} = 3x + 6y$$

14. If $x = f_1(t)$, $x = f_2(t)$ and $y = g_1(t)$, $y = g_2(t)$ be two solutions of the homogeneous linear system

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y$$

$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y$$

then write down its solution.

15. Consider the linear system

$$\frac{dx}{dt} = 5x + 3y$$

$$\frac{dy}{dt} = 4x + y$$

Show that $x = 3e^{7t}$, $x = e^{-t}$ and $y = 2e^{7t}$, $y = -2e^{-t}$ are the solutions of this system.

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SECTION—B

Answer any *five* questions : 6×5=30

16. (a) Form the PDE by eliminating the arbitrary function from the relation

$$z = f(2x+3y) + g(2x+y)$$ 4

(b) Solve : 2

$$\frac{\partial^2 z}{\partial x^2} = \cos x$$

17. (a) Find the singular solution of the PDE

$$z = px + qy + p^2 + q^2$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. 4

(b) Find the complete integral of $q + \sin p = 0$, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. 2

18. Reduce the equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = u$$

into canonical form and hence find its general solution.

19. Solve by the method of separation variables

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u, \; u(x, 0) = 6e^{-3x}$$

20. Reduce the PDE

$$\frac{\partial^2 z}{\partial x^2} - x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

into canonical form and hence solve it.

21. Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = x^3 y + e^{2x}$

22. Solve initial boundary value problem

$$u_t = 3u_{xx}, \; u(x, 0) = 17 \sin \pi x, \; u(0, t) = u(4, t) = 0$$

23. (a) Find all eigenvalues and eigen functions of eigen problem

$$\frac{\partial^2 y}{\partial x^2} + \lambda y = 0, \; y = y(x), \; 0 < x < 1, \; y(0) = y(1) = 0$$ 3

(b) Solve $2u_x + 3u_y = 0$, $u(x, 0) = \sin x$. 3

24. Solve the following system of ordinary differential equations :

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$$

$$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2$$

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25. Using operator method, find the general solution of the following linear system :

$$\frac{dx}{dt} + \frac{dy}{dt} - x - 2y = 2e^{t}$$

$$\frac{dx}{dt} + \frac{dy}{dt} - 3x - 4y = e^{2t}$$

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