

**2020/TDC (CBCS)/ODD/SEM/
MTMHCC-301T/327**

**TDC (CBCS) Odd Semester Exam., 2020
held in March, 2021**

MATHEMATICS

(3rd Semester)

Course No. : MTMHCC-301T

(Theory of Real Functions)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

*The figures in the margin indicate full marks
for the questions*

SECTION—A

1. Answer any ten of the following questions :

2×10=20

(a) Using $\varepsilon - \delta$ definition, prove that

$$\lim_{x \rightarrow 2} (5x + 7) = 17$$

(b) Show that

$$\lim_{x \rightarrow 0} \sin \frac{1}{x}$$

does not exist.

(2)

- (c) Show that

$$\lim_{x \rightarrow 0} \sin \frac{1}{x^2} = \infty$$

- (d) Define limits at infinity.

- (e) State the sequential criterion for continuity of a function.

- (f) Examine the continuity of the function
- $f: \mathbb{R} \rightarrow \mathbb{R}$
- defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

at every rational point in \mathbb{R} .

- (g) How can you 'remove' the discontinuity of the function

$$f(x) = \begin{cases} x+1, & x > 1 \\ 1, & x = 1 \\ 3-x, & x < 1 \end{cases}$$

at $x=1$?

- (h) Check if there exists a zero of
- $f(x) = x^3 - x - 2$
- , not exceeding 2.

- (i) Check the differentiability of
- $f(x) = |x+a| \forall x \in \mathbb{R}$
- at
- $x=-a$
- .

(3)

- (j) State Rolle's theorem.

- (k) Use definition to find the derivative of the function
- $f(x) = \sqrt{x}$
- ,
- $x > 0$
- .

- (l) Show that differentiability of a function
- $f: \mathbb{R} \rightarrow \mathbb{R}$
- implies continuity of
- f
- .

- (m) Show that the function
- $f: (-2, 2) \rightarrow \mathbb{R}$
- , defined by
- $f(x) = x^3 \forall x \in (-2, 2)$
- is uniformly continuous in
- $(-2, 2)$
- .

- (n) Show that the function
- $f: (0, \infty) \rightarrow \mathbb{R}$
- defined by
- $f(x) = \frac{1}{x} \forall x \in (0, \infty)$
- is not uniformly continuous.

- (o) What is a Lipschitz function? Explain with example.

- (p) Mention the non-uniform continuity criterion using sequences.

- (q) Using Lagrange's mean value theorem, show that
- $e^x \geq 1+x \forall x \in \mathbb{R}$
- .

- (r) State Taylor's theorem.

(4)

- (s) Write the Taylor expansion of $f(x) = \sin x$ at $x = 0$.
- (t) Give an example of a function which is strictly monotone increasing but not continuous.

SECTION—B

Answer any five of the following questions : $10 \times 5 = 50$

2. (a) Using $\varepsilon - \delta$ definition, show that

$$\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}, \quad c > 0 \quad 5$$

- (b) Prove the equivalence of $\varepsilon - \delta$ definition and sequential definition of limit of a function at a point. 5

3. (a) Using $\varepsilon - \delta$ definition, show that

$$\lim_{x \rightarrow 1} \frac{x}{1+x} = \frac{1}{2} \quad 5$$

- (b) If $A \subseteq \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then show that f is bounded on some neighbourhood of c . 5

(5)

4. (a) Show that a function continuous on a closed and bounded interval $[a, b]$ is itself bounded therein. 5

- (b) If f and g are continuous at $x = c$, show that the product function fg is continuous at $x = c$. 5

5. (a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and let I be an interval. Show that $f(I)$ is an interval. 5

- (b) Find all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(r) = 0$ for every rational number r . 5

6. (a) If f and g are differentiable at $x = c$ and $g(c) \neq 0$, then show that the quotient f/g is differentiable at $x = c$. 5

- (b) State and prove Lagrange's mean value theorem. 1+4=5

7. (a) Let $f : I \rightarrow \mathbb{R}$ be differentiable on the interval I . Then show that f is increasing on I iff $f'(x) \geq 0 \forall x \in I$. 5

- (b) State and prove Darboux's theorem. 1+4=5

(6)

8. (a) Show that if f and g are uniformly continuous on $A \subseteq \mathbb{R}$, then $f+g$ is uniformly continuous on A . 5

- (b) If f is uniformly continuous on $A \subseteq \mathbb{R}$ and $|f(x)| \geq k > 0 \forall x \in A$, show that $1/f$ is uniformly continuous on A . 5

9. (a) Every Lipschitz function is uniformly continuous. Can we also say that every uniformly continuous function is Lipschitz? Justify your answer. 5

- (b) Show that $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \frac{1}{1+x^2} \forall x \in \mathbb{R}$$

is uniformly continuous. 5

10. (a) State and prove Cauchy's mean value theorem. 5

- (b) Using Taylor's theorem, show that

$$1 - \frac{x^2}{2} \leq \cos x \quad \forall x \in \mathbb{R} \quad 5$$

(7)

11. (a) Derive the Taylor's series expansion of $f(x) = \ln(1+x)$ about any point $x=c$ in its domain. 5

- (b) Show that if $x > 0$, then

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2} \quad 5$$

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