2019/TDC/ODD/SEM/MTMHCC–301T/175

TDC (CBCS) Odd Semester Exam., 2019

MATHEMATICS

(3rd Semester)

Course No. : MTMHCC–301T

(Theory of Real Functions)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

(a) If $\underset{x\to c}{\text{Lt}}\, f(x)=l$, then prove that

$$\underset{x\to c}{\text{Lt}}\, |f(x)|=|l|$$

(b) Prove that

$$\underset{x\to 1}{\text{Lt}}\, x^{\frac{1}{1-x}}=\frac{1}{e}$$

(c) State sequential criterion for limits.

2. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] :

(a) If

$$\lim_{x\to 0} \frac{x^a \sin^b x}{\sin x^c}$$

where $a, b, c \in R - \{0\}$ exist and has non-zero values, then show that $a+b=c$. 3

(b) *(i)* If $f(x) \le g(x) \le h(x)$ in a certain neighbourhood of the point c and

$$\operatorname*{Lt}_{x\to c} f(x) = l = \operatorname*{Lt}_{x\to c} h(x)$$

then prove that

$$\operatorname*{Lt}_{x\to c} g(x) = l$$ 3

(ii) Prove that the limit of a function, if it exists, is unique. 4

(c) If

$$\operatorname*{Lt}_{x\to a} f(x) \text{ and } \operatorname*{Lt}_{x\to a} g(x)$$

exist finitely, then prove that

$$\operatorname*{Lt}_{x\to a} \{f(x)\cdot g(x)\} = \operatorname*{Lt}_{x\to a} f(x)\cdot \operatorname*{Lt}_{x\to a} g(x)$$ 4

(d) *(i)* Evaluate : 3

$$\operatorname*{Lt}_{x\to \frac{1}{3}+} x\left[\frac{1}{x}\right]$$

(ii) Using definition of limit, show that

$$\operatorname*{Lt}_{x\to 0} x^2 \sin\frac{1}{x} = 0$$ 3

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) Give an example of a bounded function which is discontinuous at every point of its domain.

(b) What must be the value of $f(0)$ so that $f(x) = (x+1)^{\cot x}$ becomes continuous at $x=0$?

(c) Define the following terms :

(i) Removable discontinuity

(ii) Discontinuity of first kind

4. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] :

(a) Prove that a function f defined on an interval I is continuous at a point c in I if and only if for any sequence $< c_n >$ in I converging to c the sequence $\langle f(c_n)\rangle$ converging to $f(c)$, i.e.,

$$c_n \to c \Rightarrow \langle f(c_n)\rangle \to f(c)$$

as $n\to\infty$. 5

(b) Show that the function

$$f(x) = |x| + |x-1| + |x-2|$$

is continuous at points $x = 0, 1, 2$. 5

(c) State and prove intermediate value theorem. 4

(d) (i) Show that the function f defined on R

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every point. 3

(ii) Show that the function f defined as $f(x) = x - [x]$, where $[x]$ denotes the integral part of x is discontinuous for all integral values of x. 3

UNIT—III

5. Answer any *two* of the following : 2×2=4

(a) State Rolle's theorem.

(b) State Darboux's theorem.

(c) Let f and g be two functions with the same domain D. Give an example to show that if fg is derivable at $C \in D$, then f and g are not necessarily so at C.

6. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) State and prove Carathéodory's theorem. 4

(b) (i) Show that the function $f(x) = x|x|$ is derivable at origin. 3

(ii) Suppose f and g are continuous in $[a, b]$ and differentiable on (a, b). If $f'(x) = g'(x)\ \forall x \in (a, b)$, then prove that there exists a constant K such that $f = g + k$ on $[a, b]$, i.e., f and g differ by a constant on $[a, b]$. 3

(c) A function f is defined on R as follows :

$$f(x) = \begin{cases} \dfrac{x(e^{\frac{1}{x}} - e^{-\frac{1}{x}})}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Find $Lf'(0)$ and $Rf'(0)$. Is f derivable at $x = 0$? 4

(d) (i) If a function f is derivable at a point c and $f(c) \neq 0$, then show that function $\frac{1}{f}$ is also derivable at c and

$$\left(\frac{1}{f}\right)(c) = \frac{-f'(c)}{\{f(c)\}^2}$$ 3

(ii) Prove that, if

$$\frac{a_0}{n+1}+\frac{a_1}{n}+\cdots+\frac{a_n-1}{2+a_n}=0$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real numbers, then the equation

$$a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0$$

has at least one real root between 0 and 1. 3

UNIT—IV

7. Answer any *two* of the following : 2×2=4

(a) Define uniform continuity of a function in a domain.

(b) Prove that $f(x)=\sin x$ is uniformly continuous on R.

(c) Define Lipschitz's function.

8. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Prove that a function which is continuous in a closed and bounded interval $[a, b]$ is uniformly continuous in $[a, b]$. 5

(b) If $f: A \to R$ is uniformly continuous on a subset A on R and if $\langle x_n \rangle$ is a Cauchy sequence in A, then prove that $\langle f(x_n) \rangle$ is a Cauchy sequence in R. 5

(c) (i) Show that the function

$$f(x)=x^2 \ \forall x \in R$$

is uniformly continuous on $[-1, 1]$ but not in R. 3

(ii) If $f: A \to R$ is a Lipschitz function, then f is uniformly continuous on A. 3

(d) Show that the function f defined as

$$f(x)=\begin{cases}\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$$

is not uniformly continuous on $[0, \infty)$. 4

UNIT—V

9. Answer any *two* of the following : 2×2=4

(a) State Maclaurin's theorem with Lagrange's form of remainder.

(b) State Cauchy's mean value theorem.

(c) Suppose f is continuous on $[a, b]$ and differentiable on (a, b),

$$f'(x)=0 \ \forall x \in (a, b)$$

then show that f is constant on $[a, b]$.

10. Answer *either* [(*a*) and (*b*)] *or* [(*c*) and (*d*)] :

(*a*) Using Cauchy's mean value theorem for the functions $f(x) = e^x$ and $g(x) = e^{-x}$ show that there exists a point c in (a, b) such that c is the arithmetic mean between a and b. 5

(*b*) State and prove Taylor's theorem with Lagrange's form of remainder. 5

(*c*) If f'' is continuous at $x = a$, then show that

$$\underset{h \to 0}{\text{Lt}} \left[\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \right] = f''(a)$$ 4

(*d*) (*i*) Using Taylor theorem, show that

$$x - \frac{x^6}{6} < \sin x < x, \text{ for } x > 0$$ 4

(*ii*) Explain why $\sqrt{x}$ and $x^{5/2}$ cannot be expanded in Maclaurin's infinite series. 2

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TDC (CBCS) Odd Semester Exam., 2019

MATHEMATICS

(3rd Semester)

Course No. : MTMHCC–302T

(Group Theory)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : $2 \times 2 = 4$

(a) Define group with example.

(b) What do you mean by the order of an element of a group? What is the order of an infinite group?

(c) Define a semigroup. Is it a group?

2. Answer either [(a) and (b)] or [(c) and (d)] :

(a) Show that the set M of all complex numbers z such that $|z| = 1$ form a group w.r.t. the operation of multiplication of complex numbers. Is it abelian? 4+1=5

(b) Consider the permutations

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

Compute $f * g$, $g * f$ and f^{-1}. 2+2+1=5

(c) Prove that the set of all nth roots of unity forms a finite abelian group of order n w.r.t. multiplication. 5

(d) Define symmetric group. Show that the symmetric group S_3 is non-abelian. What is the order of S_n? 1+3+1=5

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) What is the index of a subgroup of a group?

(b) Write two subgroups of $\mathbb{Z}$ under addition.

(c) Is the union of two subgroups a subgroup? Justify your answer.

4. Answer either [(a) and (b)] or [(c) and (d)] :

(a) Define inverse of a complex. If H is a subgroup of a group G, then show that $H^{-1} = H$. Also show that the converse is not true. 1+2+2=5

(b) What do you mean by the product of two subgroups of a group? Show that the product of two subgroups H and K of a group G is a subgroup iff $HK = KH$. 1+4=5

(c) What is the normalizer of an element of a group? Prove that the centre of a group G is a subgroup of G. 2+3=5

(d) Prove that the necessary and sufficient condition for a nonempty subset H of a group G to be a subgroup is that $ab^{-1} \in H$, where $a, b \in H$ and b^{-1} is the inverse of b in G. 5

UNIT—III

5. Answer any *two* of the following questions : 2×2=4

(a) What are the generators of the cyclic group $\{1, -1, i, -i\}$?

(b) Define alternating group. What is its order?

(c) What is the length of an identity permutation? Is it cyclic?

6. Answer either [(a) and (b)] or [(c) and (d)] :

(a) Define a cyclic group. How many generators are there of a cyclic group of order 8? 2+3=5

(b) When are two cyclic permutations said to be disjoint? Give an example to show that the product of two disjoint cyclic permutations on a set commute with each other. 2+3=5

(c) Prove that—

(i) every group of prime order is cyclic;

(ii) if a is a generator of a cyclic group G, then a^{-1} is also a generator of G. 3+2=5

(d) What do you mean by even and odd permutations? Give one example of each. Is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ odd or even? 2+2+1=5

UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Define a factor group.

(b) Define normal subgroup with example.

(c) Define simple group and give an example of it.

8. Answer either [(a) and (b)] or [(c) and (d)] :

(a) State and prove Lagrange's theorem. 1+4=5

(b) (i) Show that every subgroup of a cyclic group is normal. 3

(ii) Show that the factor group of an abelian group is abelian. 2

(c) Define right coset and left coset of a subgroup of a group. When are they same? Show that any two right cosets are either disjoint or identical. 1+1+3=5

(d) If H and K are two subgroups of a group G and H is normal in G, then prove that HK is a subgroup of G and $H \cap K$ is a normal subgroup of K. 5

UNIT—V

9. Answer any *two* of the following questions : 2×2=4

(a) What do you mean by group homomorphism?

(b) Show that the homomorphic image of an abelian group is abelian.

(c) Let G and G' be two groups and $f : G \to G'$ be a homomorphism. Then show that $f(a^{-1}) = [f(a)]^{-1}$, $\forall\, a \in G$.

10. Answer either [*(a)* and *(b)*] or [*(c)* and *(d)*] :

(a) Define kernel of a homomorphism. If $f : G \to G'$ be a homomorphism, then show that kernel of f is a normal subgroup of G. 1+4=5

(b) State and prove Cayley's theorem. 1+4=5

(c) Write down the identity element of a quotient group. Show that any infinite cyclic group is isomorphic to the group of integers under addition. 1+4=5

(d) State and prove the fundamental theorem of homomorphism. 1+4=5

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2019/TDC/ODD/SEM/
MTMHCC–303T/177

TDC (CBCS) Odd Semester Exam., 2019

MATHEMATICS

(3rd Semester)

Course No. : MTMHCC–303T

(PDE and Systems of ODE)

Full Marks : 50
Pass Marks : 20

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* questions from the following : 2×2=4

(a) Find a partial differential equation by eliminating a and b from the equation $z=(x-a)^2+(y-b)^2$.

(b) If $z=f(x-at)+F(x+at)$, show that

$$\frac{\partial^2 z}{\partial t^2}=a^2\frac{\partial^2 z}{\partial x^2}$$

(c) Solve the equation $\frac{\partial z}{\partial x}+2yz=y\sin x$

2. *(a)* Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2+z^2)=0$. What is the order of this partial differential equation? 6

Or

(b) Solve the following equations : 3+3=6

(i) $x\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial y}=4x+2y+z$

(ii) $\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial x}=2x$

UNIT—II

3. Answer any *two* of the following : 2×2=4

(a) Solve by using Lagrange's method

$$2\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}=1$$

(b) Solve by using the method of separable variables :

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$$

(c) Find the characteristics of the first order linear partial differential equation

$$x^2\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+xyz=1$$

4. *(a)* Solve : 4+2=6

(i) $x(y^2+z)\frac{\partial z}{\partial x}-y(x^2+z)\frac{\partial z}{\partial y}=z(x^2-y^2)$

(ii) $yz\frac{\partial z}{\partial x}+zx\frac{\partial z}{\partial y}=xy$

Or

(b) *(i)* Solve by using the method of separation of variables : 3

$$\frac{\partial^2 z}{\partial x \partial y}+9x^2y^2z^2=0$$

(ii) Find the integral surface of the partial differential equation : 3

$$4yz\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}+2y=0$$

Passing through the curve $y^2+z^2=1,\ x+z=2$.

UNIT—III

5. Answer any *two* of the following : 2×2=4

(a) Solve $(D^2+2DD'+D'^2)z=0$, where $D\equiv\frac{\partial}{\partial x}$ and $D'\equiv\frac{\partial}{\partial y}$.

(b) Find the particular integral of the partial differential equation

$$(D^2 + D'^2)z = x + y$$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial y}$.

(c) Find the characteristics of the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

6. (a) Find the canonical form of the partial differential equation : 6

$$\frac{\partial^2 z}{\partial x^2} - 9\frac{\partial^2 z}{\partial y^2} = 0$$

and hence solve it.

Or

(b) Solve :

$$(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy,$$

where $D \equiv \frac{\partial}{\partial x}$ and $D' \equiv \frac{\partial}{\partial z}$ 6

UNIT—IV

7. Answer any *two* of the following : 2×2=4

(a) Solve by using the method of separation of variables :

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0,\ u(x,0) = 4e^{-x}$$

(b) Give interpretation of the variables x, t and u in the heat conduction equation

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$$

(c) Define homogeneous and non-homogeneous boundary condition of a second order partial differential equation.

8. (a) Derive wave equation on a stretched string. 6

Or

(b) Solve the following initial boundary value problem of one-dimensional heat conduction equation by the method of separation of variables : 6

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2},\ 0 < n < L,\ 0 < t < \infty$$

$$u(x, 0) = f(x),\ 0 < x < L$$

$$u(0, t) = u(L, t) = 0,\ 0 < t < \infty$$

UNIT—V

9. Answer any *two* of the following : 2×2=4

(a) Define a differential operator.

(b) What is the normal form of a homogenous system of linear ordinary differential equation?

(c) What is the matrix form of a system of non-homogeneous system of linear ordinary differential equation?

10. Solve (any *one*) : 6

(a) $$\frac{dx}{dt}+2x-3y=t$$
$$\frac{dy}{dt}-3x+2y=e^{2t}$$

(b) $$\frac{dx}{dt}-7x+y=0$$
$$\frac{dy}{dt}-2x-5y=0$$

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