2021/TDC/CBCS/ODD/
MATDSC/GE–101T/324A

TDC (CBCS) Odd Semester Exam., 2021
Held in March, 2022

MATHEMATICS

(1st Semester)

Course No. : MATDSC/GE–101T

(Differential Calculus)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *twenty* of the following questions : 1×20=20

1. Define limit of a function at a point.

2. Write the value of $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$.

3. Does $\lim_{x \to 2} \frac{|x-2|}{x-2}$ exist?

4. Give an example of a function $f(x)$ such that $\lim_{x \to 0} f(x)$ and $f(0)$ exist and are equal.

5. Does $\lim_{x \to 0} \frac{1}{x}$ exist?

6. Define a continuous function.

7. Is the function $f(x) = \frac{1}{x-1}$ continuous?

8. Show that the derivative of an even function is odd function.

9. Is the function $f(x) = \sin\frac{1}{x}$ continuous at $x = 0$?

10. Find $\frac{d}{dx}\{\log(\sec x + \tan x)\}$.

11. If $y = \cos x \cos 2x$, find y_n.

12. If $y = \sin^{-1} x$, then show that

$$(1 - x^2)y_2 - xy_1 = 0$$

13. If $u(x, y) = x \sin y + y \sin x$, then find $\frac{\partial^2 u}{\partial x^2}$.

14. Define a homogeneous function of degree n in two variables.

15. Is the function $f(x, y) = x^2 \log\left(\frac{y}{x}\right)$ homogeneous? If so, find its degree.

16. Find the slope of the tangent to the curve $y = x^2$ at the point (1, 1).

17. What is the condition that the two curves $\phi(x, y) = 0$ and $\psi(x, y) = 0$ cut orthogonally?

18. Write the formulae for subtangent and subnormal for a plane curve in Cartesian form.

19. Define radius of curvature at any point on a curve.

20. Which axis is the curve $y^2 = x$ symmetrical about?

21. Evaluate $\lim_{x \to \frac{\pi}{2}} (1 - \sin x) \tan x$.

22. State Rolle's theorem.

23. Write Cauchy's form of remainder term in Taylor's theorem.

24. What do you mean by the maximum or minimum value of a function $f(x)$ at $x = c$?

25. Write the geometrical meaning of Lagrange's mean value theorem.

SECTION—B

Answer any *five* of the following questions : 2×5=10

26. Using ε-δ definition, show that

$$\underset{x \to 2}{\text{Lt}} (5x - 4) = 6$$

27. State Cauchy's necessary and sufficient condition for the existence of limit of a function at a point.

28. Show that if a function $f(x)$ is differentiable at a point $x = a$, then it is also continuous at $x = a$.

29. Prove that if a function f is continuous, then $|f|$ is also continuous.

30. Find the nth derivative of $x^{n-1} \log x$.

31. If $v = z \tan^{-1} \dfrac{y}{x}$, then show that $v_{xx} + v_{yy} + v_{zz} = 0$.

32. Find the equation of the tangent at (x, y) to the curve $\left(\dfrac{x}{a}\right)^{2/3} + \left(\dfrac{y}{b}\right)^{2/3} = 1$.

33. Find the radius of curvature at the point (s, ψ) of the curve

$$s = a \sec \psi \tan \psi + a \log(\sec \psi + \tan \psi)$$

34. If $f(h) = f(0) + hf'(0) + \dfrac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$, then find θ when $h = 1$ and $f(x) = (1 - x)^{\frac{5}{2}}$.

35. Evaluate $\lim\limits_{x \to 0} (\cos x)^{\frac{1}{x^2}}$.

SECTION—C

Answer any *five* of the following questions : 8×5=40

36. *(a)* Show by using Cauchy's criterion that $\underset{x \to 0}{\text{Lt}} \cos \dfrac{1}{x}$ does not exist. 4

(b) Evaluate

$$\lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \cdots + \frac{n}{n^2} \right)$$

and $\lim\limits_{n \to \infty} \dfrac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3}$ 2+2=4

37. *(a)* Show that $\lim_{x \to 2} [x]$ does not exist, where $[x]$ denotes the integral part of x. 2

(b) If $f(x) = ax^2 + bx + c$, then show that $\lim_{n \to 0} \frac{f(x+n) - f(x)}{n} = 2ax + b$. 2

(c) If $\phi(x) = \frac{(x+2)^2 - 4}{x}$, then show that $\lim_{x \to 0} \phi(x) = 4$, although $\phi(0)$ does not exist. 2

(d) Show that $\lim_{x \to 0} \frac{1}{2 + e^{\frac{1}{x}}}$ does not exist. 2

38. *(a)* Show that the function $f(x) = |x - 1|$ is not differentiable at $x = 1$, though it is continuous there. 4

(b) Examine the differentiability of the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

at $x = 0$. 4

39. *(a)* Find the values of a and b such that the function

$$f(x) = \begin{cases} x + \sqrt{2} a \sin x, & 0 \leq x \leq \frac{\pi}{4} \\ 2x \cot x + b, & \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ a \cos 2x - b \sin x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

is continuous for all values of x in the interval $0 \leq x \leq \pi$. 4

(b) Let f be a function such that for all real values of x, y, $f(x + y) = f(x) + f(y)$. If f is continuous at a point $x = a$, then prove that f is continuous for all real values of x. 4

40. *(a)* If $u = \sin ax + \cos ax$, then show that

$$u_n = a^n \left\{1 + (-1)^n \sin 2ax\right\}^{1/2}$$ 3

(b) If $x = \sin t$, $y = \sin kt$, where k is a constant, then show that

$$(1 - x^2) y_2 - x y_1 + k^2 y = 0$$ 3

(c) If $u = xyf\left(\frac{y}{x}\right)$, then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$ 2

41. *(a)* State and prove Leibnitz's theorem on successive differentiation. 4

(b) State and prove Euler's theorem on homogeneous function of degree n in two variables x and y. 4

42. *(a)* Find the condition that the conics $ax^2 + by^2 - 1 = 0$ and $a_1 x^2 + b_1 y^2 - 1 = 0$ shall cut orthogonally. 3

(b) Find the length of the Cartesian subtangent of the curve $y = e^{-\frac{x}{2}}$. 2

(c) Show that for the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the radius of curvature at an extremity of the major axis is equal to half of the latus rectum. 3

43. (a) Prove that all the points of the curve $y^2=4a\{x+a\sin(x/a)\}$ at which the tangent is parallel to the x-axis lie on a parabola. 3

(b) Find the length of the polar subtangent for the curve $r=a(1+\cos\theta)$ at $\theta=\frac{\pi}{2}$. 2

(c) If ρ_1 and ρ_2 be the radii of curvature at the end points of a focal chord of the parabola $y^2=4ax$, then show that

$$\rho_1^{-\frac{2}{3}}+\rho_2^{-\frac{2}{3}}=(2a)^{-\frac{2}{3}}$$ 3

44. (a) State and prove Lagrange's mean value theorem. 4

(b) Show that the maximum value of $x^2\log\left(\frac{1}{x}\right)$ is $\frac{1}{2e}$. 4

45. (a) Find a and b such that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$$ 4

(b) State and prove Cauchy's mean value theorem. 4

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