**2020/TDC(CBCS)/ODD/SEM/
MTMHCC–102T/325**

**TDC (CBCS) Odd Semester Exam., 2020
held in March, 2021**

MATHEMATICS

(1st Semester)

Course No. : MTMHCC–102T

(Higher Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

1. Answer any *ten* of the following questions : 2×10=20

(a) Apply De Moivre's theorem to prove that

(i) $\cos 2x = \cos^2 x - \sin^2 x$

(ii) $\sin 2x = 2\sin x \cos x$

(b) Expand $\sin^3 x$ in ascending powers of x.

(c) Show that i^i is purely real.

(d) Show that

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \cdots\right)$$

(e) Check if the relation R on $\mathbb{Z}$ defined by $(a, b) \in R$ iff $|a-b| = 0$ or 5 is an equivalence relation.

(f) If $f: X \to Y$ and $g: Y \to Z$ such that $g \circ f: X \to Z$ is onto, then show that g is onto.

(g) Check if $f: \mathbb{R} - \{1\} \to \mathbb{R}$ defined by $f(x) = \frac{x+1}{x-1}$ $\forall x \in \mathbb{R} - \{1\}$ is invertible.

(h) Show that the set of odd natural numbers is countably infinite.

(i) Find the quotient and remainder in the division of −315 by 4 and in the division of 315 by −4.

(j) Prove or disprove if $a / b + c$, then either a / b or a / c.

(k) Find the remainder when 3^{2021} is divided by 8.

(l) Prove Euclid's lemma :

If a / bc with gcd $(a, b) = 1$, then a / c

(m) Find the maximum number of +ve roots of the equation

$$x^3 - 2x^2 + 3x + 7 = 0$$

(n) If the sum of two roots of the equation $x^3 + px^2 + qx + r = 0$ is zero, find the third root.

(o) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of $\sum \frac{1}{\alpha}$

(p) Write down the equation whose roots are reciprocal of the roots of the equation $2x^3 + 3x^2 + 4x + 5 = 0$.

(q) What do you mean by canonical form of matrices?

(r) Define rank of a matrix.

(s) Prove that every singleton set containing non-zero vector is LI.

(t) Show that the set

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), \}$$

is not LI.

SECTION—B

Answer *any* **five** questions

2. *(a)* State De Moivre's theorem and prove it for positive integral index. 4

(b) If

$$x = \frac{2}{\lfloor 1} - \frac{4}{\lfloor 3} + \frac{6}{\lfloor 5} - \frac{8}{\lfloor 7} + \cdots \text{ to } \infty$$

and $y = 1 + \frac{2}{\lfloor 1} - \frac{2^3}{\lfloor 3} + \frac{2^5}{\lfloor 5} - \cdots$ to ∞

then show that $x^2 = y$. 3

(c) Show that

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i(2n\pi + \tan^{-1} y/x)$$ 3

3. *(a)* If $x = \cos\theta + i\sin\theta$ and $1+\sqrt{1-a^2} = na$, then prove that

$$1 + a\cos\theta = \frac{a}{2n}(1+nx)(1+\frac{n}{x}).$$ 3

(b) Express $(\alpha + i\beta)^{p+iq}$ in the form of $A + i\beta$. 4

(c) If $x < (\sqrt{2}-1)$, then prove that

$$2(x - x^3/3 + x^5/5 \cdots) = \frac{2x}{1-x^2} - \frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3 + \frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5 \cdots$$ 3

4. *(a)* Show that the relation R on $\mathbb{Z}$ defined by $(a, b) \in R$ iff $7/a-b$ is an equivalence relation on $\mathbb{Z}$. What are the distinct equivalence classes in $\mathbb{Z}$ under this relation? 3+2=5

(b) Show that $f: x \to y$ is invertible iff $\exists$ a function $g: y \to x$ such that $g \circ f = I_X$ and $f \circ g = I_Y$, where I_X and I_Y are the identity functions on X and Y respectively. 5

5. *(a)* Give example, with justification, of a relation that is symmetric, transitive but not reflexive. 2

(b) If $f: X \to Y$ is invertible, show that $(f^{-1})^{-1} = f$. 3

(c) Show that the set of rational numbers is countable. 5

6. *(a)* State and prove division algorithm. 5

(b) Using mathematical induction, prove that $24 / 2 \cdot 7^n + 3 \cdot 5^n - 5$. 5

7. *(a)* Use Euclidean algorithm to obtain integers x and y satisfying

$$\gcd(1769, 2378) = 1769x + 2378y$$ 3

(b) Find all primes which are of the form $n^3 - 1$. 2

(c) Let $n \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, show that—

(i) if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac = bd \pmod{n}$;

(ii) if $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$ for any +ve integer k. 2½+2½=5

8. (a) Show that the equation

$$x^9 + 5x^8 - x^3 + 7x + 2 = 0$$

has at least four imaginary roots. 3

(b) If the roots of

$$x^3 + 3px^2 + 3qx + r = 0$$

are in HP, prove that $2q^3 = r(3pq - r)$. 3

(c) Solve the equation $x^3 - 12x + 65 = 0$ by Cardan's method. 4

9. (a) Solve the equation $x^3 - 7x^2 + 36 = 0$; given that one root is double of another. 3

(b) If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, find $\sum \alpha^2 \beta\gamma$ in terms of p, q, r, s. 4

(c) If α, β, γ are roots of the equation $x^3 - px^2 + qx - r = 0$ then find the equation whose roots are $\beta\gamma + \frac{1}{\alpha}, \gamma\alpha + \frac{1}{\beta}, \alpha\beta + \frac{1}{\gamma}$. 3

10. (a) Show that the rank of the transpose of a matrix is the same as that of the original matrix. 5

(b) Reduce the following matrix to normal form : 5

$$\begin{pmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

11. (a) Find the rank of the matrix

$$\begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$$ 5

(b) Solve the following system of equations by Gaussian elimination method : 5

$$x + 2y + 3z = 10$$
$$2x - 3y + z = 1$$
$$3x + y - 2z = 9$$

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