2019/TDC/ODD/SEM/MTMHCC–101T/172

TDC (CBCS) Odd Semester Exam., 2019

MATHEMATICS

(1st Semester)

Course No. : MTMHCC–101T

(Calculus)

Full Marks : 50
Pass Marks : 20

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following : 2×2=4

(a) Find y_n, if $y = \sin 3x \cos 2x$.

(b) If $y = a\cos(\log x) + b\sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$.

(c) Using Leibnitz's theorem, differentiate n times the equation

$$(1+x^2)y_2 + (2x-1)y_1 = 0$$

2. Answer *either* (a) and (b) *or* (c) and (d) :

(a) If $y = x^{2n}$, where n is a +ve integer, show that

$$y_n = 2^n \{1\cdot 3\cdot 5\cdots(2n-1)\}\, x^n \qquad 3$$

(b) If $y = \tan^{-1} x$, then prove that

$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)_{n-1} = 0$$

Find also the value of $(y_n)_0$. 3

(c) If $y = e^{-x}\cos x$, prove that

$$y_4 + 4y = 0 \qquad 3$$

(d) By forming in two different ways the nth derivative of x^{2n}, show that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2\cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2} + \cdots = \frac{(2n)!}{(n!)^2} \qquad 3$$

UNIT—II

3. Answer any *two* of the following : 2×2=4

(a) Evaluate :

$$\underset{n\to\infty}{\mathrm{Lt}} \frac{x^4}{e^x}$$

(b) Define inflection point on a curve. Give example to show that a point where $y^n = 0$ is not necessarily an inflection point.

(c) Write the parametric and Cartesian equation of an astroid and draw a rough sketch.

4. Answer *either* (a) and (b) *or* (c) and (d) :

(a) Find—

(i) $\underset{x\to 0}{\mathrm{Lt}} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$;

(ii) $\underset{x\to \pi/2}{\mathrm{Lt}} (1-\sin x)\tan x$. 1½+1½=3

(b) Examine the curve $y = -2x^3 + 6x^2 - 3$ for concavity and points of inflection, if any. Also draw a rough sketch. 3

(c) Evaluate : 1½+1½=3

(i) $\underset{x\to 0}{\mathrm{Lt}} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}}$

(ii) $\underset{x\to 1}{\mathrm{Lt}} \left(\frac{1}{x^2-1} - \frac{2}{x^4-1}\right)$

(d) Define horizontal asymptote of a curve. Find the asymptotes of the curve

$$y = \frac{x+3}{x+2} \qquad 3$$

UNIT—III

5. Answer any *two* of the following : 2×2=4

(a) Obtain a reduction formula for

$$\int (\log x)^n \, dx$$

(b) Evaluate :

$$\int_0^{\pi/2} \cos^8 \theta \, d\theta$$

(c) If

$$I_n = \int_0^{\pi/2} \sin^n x \, dx$$

n being a +ve integer, show that

$$I_n = \frac{n-1}{n} I_{n-2}$$

6. Answer *either* (a) and (b) *or* (c) and (d) :

(a) If

$$I_{m,n} = \int \sin^m x \cos nx \, dx$$

show that

$$(n^2 - m^2) I_{m,n} = \sin^{m-1} x (n \sin nx \sin x + m \cos nx \cos x) - m(m-1) I_{m-2,n}$$ 3

(b) If

$$U_n = \int_0^{\pi/2} x^n \sin x \, dx, \quad n > 0$$

show that

$$U_n + n(n-1) U_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$ 3

(c) Obtain a reduction formula for

$$I_{m,n} = \int \cos^m x \cos nx \, dx$$

connecting with $I_{m-1, n-1}$. 3

(d) Prove that

$$\int_0^{\pi/2} \sin^n x \, dx$$

$$= \frac{(n-1)(n-3)\cdots 4\cdot 2}{n(n-2)\cdots 5\cdot 3}, \text{ if } n \text{ is odd}$$

$$= \frac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2}\cdot\frac{\pi}{2}, \text{ if } n \text{ is even}$$

3

UNIT—IV

7. Answer any *two* of the following : 2×2=4

(a) What do you mean by rectification of plane curve? Write the formula to find the length of the curve $y = f(x)$ from $x = a$ to $x = b$.

(b) Determine the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

(c) Find the surface area of a solid generated by revolving the semicircular arc of radius c about the axis of x.

8. Answer *either* (a) and (b) *or* (c) and (d) :

(a) Show that the perimeter of the curve $r = a(1-\cos\theta)$ is $8a$. 3

(b) Show that the area cut-off a parabola by any double ordinate is two-third of the corresponding rectangle contained by the double ordinate and its distance from the vertex. 3

(c) Find the area of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$ 3

(d) Find the surface area of the solid generated by revolving the cycloid $x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$ about its base. 3

UNIT—V

9. Answer any *two* of the following : 2×2=4

(a) Show that $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar.

(b) Find the vector equation of the line parallel to the vector $\hat{i}$ and passing through the point (0, 1, 0).

(c) Show that the derivative of a constant vector is zero.

10. Answer *either* (a) and (b) *or* (c) and (d) :

(a) Prove that $[\vec{a}\times\vec{b} \;\; \vec{b}\times\vec{c} \;\; \vec{c}\times\vec{a}]$

$$= \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{b} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= [\vec{a}\;\vec{b}\;\vec{c}]^2$$ 3

(b) Find the vector equation of a plane passing through three given points. 3

(c) If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, where t is a scalar, show that

$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = 216$$

3

(d) If $\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} + \hat{j} - \hat{k}$ are position vectors of the extremities of a diameter of a sphere, find the equation of the sphere both in vector and Cartesian forms. 3

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2019/TDC/ODD/SEM/MTMHCC–102T/173

TDC (CBCS) Odd Semester Exam., 2019

MATHEMATICS

(1st Semester)

Course No. : MTMHCC–102T

(Higher Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* from the following questions : 2×2=4

(a) Expand $\cos^2 \theta$ in the powers of θ.

(b) If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, then prove that

$$x_1 x_2 \cdots \text{ to infinity} = -1$$

(c) Show that

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5}\cdot\frac{1}{3^2} - \frac{1}{7}\cdot\frac{1}{3^3} + \cdots\right)$$

2. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x + y + z = xyz$, then prove that

$$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) + 1 = 0$$ 5

(b) (i) Show that the product of n, nth roots of unity is $(-1)^{n-1}$. 3

(ii) Show that

$$\frac{\pi}{12} = \left(1 - \frac{1}{3^{1/2}}\right) - \frac{1}{3}\left(1 - \frac{1}{3^{3/2}}\right) + \frac{1}{5}\left(1 - \frac{1}{3^{5/2}}\right) - \cdots \infty$$ 2

(c) Prove that the principal value of $(\alpha + i\beta)^{x+iy}$ is wholly real or wholly imaginary according as $\frac{1}{2}y\log(\alpha^2 + \beta^2) + x\tan^{-1}\frac{\beta}{\alpha}$ is an even or odd multiple of $\frac{\pi}{2}$. 5

(d) If x_1, x_2, x_3, x_4 are the roots of the equation

$$x^4 - x^3\sin 2\alpha + x^2\cos 2\alpha - x\cos\alpha - \sin\alpha = 0$$

then show that

$$\sum \tan^{-1} x_1 = n\pi + \frac{\pi}{2} - \alpha$$ 5

UNIT—II

3. Answer any *two* from the following questions : 2×2=4

(a) Prove that inverse of function if it exists is unique.

(b) Give an example of a relation which is symmetric and transitive but not reflexive.

(c) Define the following :

(i) Equivalence class

(ii) Portion of a set

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Define bijective mapping. Show that $f : N \to N$ defined by

$$f(x) = \begin{cases} x+1, \text{ if } x \text{ is odd} \\ x-1, \text{ if } x \text{ is even} \end{cases}$$

is bijective. 5

(b) Let $f : Z \to Z$ be defined by $f(n) = 3n$ for all $n \in Z$ and $g : Z \to Z$ be defined by

$$g(n) = \begin{cases} \frac{n}{3}, & \text{if } n \text{ is a multiple of 3} \\ 0, & \text{if } n \text{ is not a multiple of 3} \end{cases}$$

$\forall n \in Z$. Show that

$$g \circ f = I_Z \text{ and } f \circ g \neq I_Z$$ 5

(c) Consider $f : R \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$, where R^+ is the set of all non-negative real numbers. Show that f is invertible with

$$f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$$ 4

(d) Let $f : X \to Y$ be a map such that $A, B \subseteq X$. Then show that

(i) $f(A \cup B) = f(A) \cup f(B)$

(ii) $f(A \cap B) \subseteq f(A) \cap f(B)$

The equality hold provided f is 1–1 map. 3+3=6

UNIT—III

5. Answer any *two* from the following questions : 2×2=4

(a) If $a \equiv b (\text{mod } m)$, then prove that

$$a^K \equiv b^K (\text{mod } m) \; \forall \; K \geq 1$$

(b) Find the remainder when 5^{48} is divided by 24.

(c) Prove that one of every three consecutive integers is divisible by 3.

6. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) State and prove division algorithm. 1+4=5

(b) If a and b are any two integers not both zero, then gcd(a, b) exists and is unique. 5

(c) Prove that well-ordering principle is equivalent to the principle finite induction. 4

(d) (i) If a, b and c are integers such that $ac \equiv bc (\text{mod } m)$, $m > 0$ is a fixed integer and $d = (c, m)$, then show that $a \equiv b\left(\text{mod } \frac{m}{d}\right)$. 3

(ii) Prove that the product of any three consecutive integers is divisible by $\lfloor 3$. 3

UNIT—IV

7. Answer any *two* from the following questions : 2×2=4

(a) Find the equation whose roots are double the roots of

$$x^3 - 6x^2 + 11x - 6 = 0$$

(b) Remove the second term of the equation

$$x^4 + 10x^3 + 26x^2 + 10x + 1 = 0$$

(c) State Descartes' rule of signs.

8. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) If one root of the equation $x^3 + px^2 + qx + r = 0$ equals the sum of the other two, then prove that

$$p^3 + 8r = 4pq$$ 5

(b) Solve by Cardan's method :

$$x^3 - 18x - 35 = 0$$ 5

(c) If α, β, γ be the roots of the equation $x^3 + 2x^2 + 3x + 4 = 0$, then find the equation whose roots are

$$\alpha - \frac{1}{\beta\gamma}, \beta - \frac{1}{\gamma\alpha}, \gamma - \frac{1}{\alpha\beta}$$ 4

(d) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n x^n + p_n = 0$$

$p_n \neq 0$, then find the value of

(i) $\sum \frac{\alpha_1^2 + \alpha_2^2}{\alpha_1 \alpha_2}$

(ii) $\sum \frac{\alpha^1}{\alpha_2^2}$

3+3=6

UNIT—V

9. Answer any *two* from the following questions : 2×2=4

(a) Prove that any subset of LI set of vectors is LI.

(b) Define linearly independent and linearly dependent set of vectors.

(c) Define echelon form of a matrix.

10. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Prove that the rank of product of two matrices cannot exceed the rank of either matrix. 5

(b) Find the rank of the matrix

$$\begin{pmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$$ 5

(c) Solve by Gaussian elimination method :

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$ 5

(d) (i) Show that the vectors $(1, 2, -3, 4)$, $(3, -1, 2, 1)$ and $(1, -5, 8, -7)$ of $R^4(R)$ are linearly dependent. 3

(ii) Show that the vectors $(2, 3, 4)$, $(0, 5, 6)$ and $(0, 0, 8)$ are linearly independent. 2

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