2024/TDC (CBCS)/EVEN/SEM/
MTMHCC-602T/238

TDC (CBCS) Even Semester Exam., 2024

MATHEMATICS

(6th Semester)

Course No. : MTMHCC-602T

(Linear Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* questions : 2×2=4

(a) Show that the set $L = \{(x, y) \mid 2x + 3y = 0\}$ is a subspace of $\mathbb{R}^2(\mathbb{R})$.

(b) Check if the vectors (1, 0, 0), (1, 0, 1) and (1, 1, 1) are linearly independent.

(c) If $\{v_1, v_2, \ldots, v_k\}$ are linearly dependent vectors in a vector space $V(\mathbb{R})$, show that one of the vectors is a linear combination of the remaining vectors.

2. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Show that the intersection of finite number of subspaces of a vector space is also a subspace of the vector space. Give example to show that the same is not true for union. 4

(b) Let V be a finite dimensional vector space. Show that any two bases of V have the same number of elements. 6

(c) If W_1 and W_2 are subspaces of finite dimensional vector, show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$ 5

(d) Find a basis and the dimension of each of the following subspaces of $\mathbb{R}^n$:

$$V = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + x_2 + \ldots + x_n = 0\}$$

$$W = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = x_2 = x_3\}$$

where $n \geq 3$. 5

UNIT—II

3. Answer any *two* questions : 2×2=4

(a) Define linear transformation from a vector space to another. Give an example, with justification, of a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$.

(b) Justify True or False :

If S is a linearly independent set in a vector space V and $T: V \to V$ is a linear transformation, then $T(S)$ is also linearly independent.

(c) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is such that $T(1, 0) = (2, 3)$, $T(0, 2) = (3, 2)$, find $T(x, y)$ for any $(x, y) \in \mathbb{R}^2$.

4. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) If $T: V \to W$ be a linear map, show that the kernel of T and range of T are subspaces of V and W respectively. 6

(b) Consider $T: \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T(x, y, z) = (z, y, x, x + y + z)$$

Write the matrix of T w.r.t. the standard ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^4$. 4

(c) State and prove the rank-nullity theorem. 5

(d) Find the rank and nullity of $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x, y, x + y)$. Also, find the rank of the matrix of T w.r.t. the standard ordered basis of $\mathbb{R}^3$. Verify whether rank of T is same as the rank of the matrix of T. 3+2=5

UNIT—III

5. Answer any *two* questions : 2×2=4

(a) If $T: V \to W$ and $S: V \to W$ are two linear transformations, show that $T + S: V \to W$ is also a linear transformation.

(b) Show that a linear map $T: V \to W$ is one-one iff $T(v) = 0 \Leftrightarrow v = 0$.

(c) Justify if there can exist any one-one linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.

6. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Let V be a finite dimensional vector space. Let $T: V \to V$ be a linear transformation. Then show that the following are equivalent : 5

(i) $\ker(T) = \{0\}$

(ii) $\operatorname{Im}(T) = V$, where $\operatorname{Im}(T)$ is the image of T

(b) Show that every n-dimensional vector space over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$. 5

(c) Let V and W be vector spaces over $\mathbb{R}$. Show that the set $L(V, W)$ of all linear transformations from V to W is a vector space with suitably defined addition and scalar multiplication. 5

(d) Let $T: V \to W$ be a linear map. Show that T is an isomorphism, if and only if, $T(B)$ is a basis of W whenever B is a basis of V. 5

UNIT—IV

7. Answer any *two* questions : 2×2=4

(a) Find the eigenvalues of the linear operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (y, x)$.

(b) If u, v and w are the eigenvectors corresponding to the eigenvalues of

$$A = \begin{pmatrix} 1 & 8 & 6 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

justify whether u, v, and w are linearly independent.

(c) If v is an eigenvector of $T: V \to V$ corresponding to λ and $\alpha \in \mathbb{R}$ is a non-zero scalar, justify if αv is also an eigenvector of T corresponding to λ.

8. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) Show that the eigenvectors of a linear operator corresponding to distinct eigenvalues are linearly independent. 5

(b) Verify that the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$$

obeys Cayley-Hamilton theorem. Hence find A^{-1}. 5

(c) Find the eigenvalues and the corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

5

(d) Define invariant subspace under a linear operator. Give an example. Show that the intersection of two invariant subspaces under a linear operator T is also invariant under T. 1+1+3=5

UNIT—V

9. Answer any *two* questions : 2×2=4

(a) Define inner product space over the field of complex numbers.

(b) Let $f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f((x_1, y_1), (x_2, y_2)) = x_1 y_1 + x_2 y_2$$

Is f an inner product? Justify your answer.

(c) Show that for any non-empty subset S of an inner product space V, the orthogonal complement $S^{\perp}$ of S is a subspace of V.

10. Answer *either* [(a) and (b)] *or* [(c) and (d)] :

(a) For any two vectors x, y in an inner product space, show that

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Give a geometrical interpretation of this identity. 4+1=5

(b) Show that $\mathbb{C}^2$ is an inner product space over $\mathbb{C}$ w.r.t. the inner product

$$\langle x, y \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$$

for $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{C}^2$. 5

(c) Let V be an inner product space. Define $d: V \times V \to \mathbb{R}$ as

$$d(x, y) = \|x - y\| \ \forall\, x, y \in V$$

Show that (V, d) is a metric space. 4

(d) Let W be a subspace of an inner product space V. Then prove that $V = W \oplus W^{\perp}$, i.e., V is an orthogonal direct sum of W and $W^{\perp}$. 6

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