2024/TDC (CBCS)/EVEN/SEM/
MTMDSC/GEC–401T/235

TDC (CBCS) Even Semester Exam., 2024

MATHEMATICS

(4th Semester)

Course No. : MTMDSC/GEC–401T

(Abstract Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *four* of the following : 1×4=4

(a) Define group.

(b) Give an example of an Abelian group.

(c) Define order of a finite group.

(d) What is the identity element of the group $(Q_+, *)$, where $a * b = (ab)/2$?

(e) Give an example of a non-Abelian group of order 8.

2. Answer any *one* of the following : 2

(a) In a group G, show that

$$(ab)^{-1} = b^{-1}a^{-1} \ \forall \ a, b \in G$$

(b) Prove that a group G is Abelian if every element of G except the identity element is of order two.

3. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] : 4×2=8

(a) Let G be a group and $a, b \in G$. Then show that the equation $ax = b$ has unique solution in G.

(b) If

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| \, a \in \mathbb{R} \right\}$$

then show that G is a commutative group under matrix multiplication.

(c) Prove that if G is an Abelian group, then for all $a, b \in G$ and all $n \in \mathbb{N}$

$$(ab)^n = a^n b^n$$

(d) Prove that nth root of unity forms an Abelian group with respect to multiplication.

UNIT—II

4. Answer any *four* of the following : 1×4=4

(a) Define cyclic group.

(b) Find all the generators of the cyclic group $(\{0, 1, 2, 3, 4, 5\}, +_6)$.

(c) Find the order of an element 3 in the cyclic group $(\{1, 2, 3, 4, 5, 6\}, \times_7)$.

(d) Define complex of a group.

(e) Define subgroup of a group.

5. Answer any *one* of the following : 2

(a) Prove that every cyclic group is an Abelian group.

(b) If H is any subgroup of group G, then show that $H^{-1} = H$.

6. Answer *either* [*(a)* and *(b)*] *or* [*(c)* and *(d)*] : 4×2=8

(a) Prove that every group of prime order is cyclic.

(b) Show that a non-empty subset H of a group G to be a subgroup of G iff

$$a, b \in H \Rightarrow ab^{-1} \in H$$

(c) Prove that every subgroup of a cyclic group is cyclic.

(d) Prove that the intersection of two subgroups of a group is again a subgroup. Is the result true for union?

UNIT—III

7. Answer any *four* of the following : 1×4=4

(a) Define left coset in a group.

(b) Let $H = \{0, \pm 3, \pm 6, \pm 9, \cdots\}$ be the subgroup of the additive group *I*. Find all right cosets of *H* in *G*.

(c) Define index of a subgroup in a group.

(d) List all subgroups of a group of order 23.

(e) Give an example of a right coset.

8. Answer any *one* of the following : 2

(a) If *a*, *b* are any two elements of a group *G*, and *H* is any subgroup of *G*, then show that $a \in Hb \Rightarrow Ha = Hb$.

(b) Use Lagrange's theorem to show that any group of prime order can have no proper subgroups.

9. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 4×2=8

(a) Prove that any two left cosets of a subgroup are either identical or disjoint.

(b) State and prove Lagrange's theorem for finite group.

(c) Show that two right cosets *Ha* and *Hb* are distinct iff two left cosets $a^{-1}H$ and $b^{-1}H$ are disjoint.

(d) Prove that the intersection of two subgroups, each of finite index, is again of finite index.

UNIT—IV

10. Answer any *four* of the following : 1×4=4

(a) Define normal subgroup of a group.

(b) Define quotient group.

(c) Define group homomorphism.

(d) A homomorphism *f* from a group *G* into a group *G′* is defined by

$$f(x) = e' \quad \forall\, x \in G$$

where *e′* is the identity in *G′*. Find ker *f*.

(e) A function $f : Z \to E$ is defined by

$$f(x) = 2x \quad \forall\, x \in Z$$

Show that *f* is a homomorphism.

11. Answer any *one* of the following : 2

(a) Show that every subgroup of an Abelian group is normal.

(b) Let f be a homomorphism from a group G into a group G'. Then show that $f(e) = e'$, where e and e' are the identities of G and G' respectively.

12. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 4×2=8

(a) Show that a subgroup H of a group G is a normal subgroup of G iff the product of two right cosets of H in G is again a right coset of H in G.

(b) State and prove fundamental theorem on homomorphism of groups.

(c) If G is a group and H is a normal subgroup of G, then show that the set G/H of all cosets of H in G is a group with respect to multiplication of cosets.

(d) Show that the kernel of a group homomorphism is a normal subgroup of the group.

UNIT—V

13. Answer any *four* of the following : 1×4=4

(a) Define ring.

(b) Give an example of a non-commutative ring.

(c) Define zero divisor.

(d) Give an example of a ring without zero divisor.

(e) Define field.

14. Answer any *one* of the following : 2

(a) If R is a ring such that

$$a^2 = a \quad \forall\, a \in R$$

then prove that $a + a = 0$.

(b) Give an example of a non-commutative ring with unity.

15. Answer *either* [(a) and (b)] *or* [(c) and (d)] : 4×2=8

(a) Prove that a ring is commutative iff

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \forall\, a, b \in R$$

(b) Prove that every finite integral domain is a field.

(c) Show that

$$(\{0, 1, 2, 3, 4, 5\}, +_6, \times_6)$$

is a ring.

(d) Show that, a ring R is without zero divisor iff the cancellation laws hold in R.

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