2024/TDC (CBCS)/EVEN/SEM/
MTMHCC–403T/234

TDC (CBCS) Even Semester Exam., 2024

MATHEMATICS

(4th Semester)

Course No. : MTMHCC–403T

(Ring Theory)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

(a) Give an example to show that union of two subrings may not be a subring.

(b) Prove that in an integral domain R (with unity) the only idempotents are the zero and unity.

(c) Define characteristic of a ring. Give an example of a ring whose characteristic is 6(six).

2. Answer *either* (a) and (b) *or* (c) and (d) :

(a) Prove that a non-zero finite integral domain is a field. 5

(b) If R is a division ring, then show that the centre $Z(R)$ of R is a field. 5

(c) If D is an integral domain and if $na = 0$ for some $0 \neq a \in D$ and some integer, $n \neq 0$, then show that the characteristic of D is finite. 4

(d) Let R be a commutative ring with unity. Show that—

(i) if $a \in R$ is a unit, then a is not nilpotent;

(ii) if $n \in R$ is nilpotent, then $1+n$ is a unit;

(iii) the sum of a nilpotent element and a unit is a unit. 6

UNIT—II

3. Answer any *two* questions : 2×2=4

(a) Prove that $6\mathbb{Z}$ is an ideal of $\mathbb{Z}$.

(b) Define prime ideal. Give an example of it.

(c) Give an example of an ideal of a ring which is neither prime nor maximal with justification.

4. Answer *either* (a) and (b) *or* (c) and (d) :

(a) Let R be a ring with unity, such that R has no right ideals except $\{0\}$ and R, then show that R is a division ring. 5

(b) Let R be a commutative ring with unity. Show that every maximal ideal of R is prime ideal. 5

(c) Let $A \neq R$ be an ideal of R, then for any $x \in R$, $x \notin A$, if $A + (x) = R$, show that A is maximal ideal of R and conversely. 5

(d) Show that a commutative ring R is an integral domain if and only if $\{0\}$ is a prime ideal. 5

UNIT—III

5. Answer any *two* questions : 2×2=4

(a) If $f : R \to R'$ be a ring homomorphism then prove that—

(i) $f(0) = 0'$

(ii) $f(-a) = -f(a)$

where $0, 0'$ are zeros of the rings R and R' respectively.

(b) State fundamental theorem of ring homomorphism.

(c) Let I be an ideal of a ring R. Show that

(i) if R is commutative then so is R/I

(ii) if R has unity 1 then $1+I$ is unity of R/I.

6. Answer *either* (a) and (b) *or* (c) and (d) :

(a) Let $\mathbb{Z}$ be the ring of integers. Show that the only homomorphisms from $\mathbb{Z} \to \mathbb{Z}$ are the identity and zero mapping. 5

(b) Let $B \subseteq A$ be two ideals of a ring R, then prove that $\frac{R}{A} \cong \frac{R/B}{A/B}$. 5

(c) Prove that any homomorphism of a field is either a monomorphism or takes each element to zero. 5

(d) Find all the ring homomorphisms from $Z_{20} \to Z_{30}$. 5

UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Let $R[x]$ be the ring of polynomials over a ring R. Then prove that if R has unity, then $R[x]$ has unity.

(b) Give an example to show that quotient ring of an integral domain may not be an integral domain.

(c) Define Euclidean domain. Give an example of it.

8. Answer *either* (a) and (b) *or* (c) and (d) :

(a) Let $R[x]$ be the ring of polynomials of a ring R. Then prove that R is an integral domain if and only if $R[x]$ is an integral domain. 5

(b) Let a, b be two non-zero elements of a Euclidean domain R. If b is not a unit in R, then prove that $d(a) < d(ab)$. 5

(c) If F is a field then prove that $F[x]$ is a Euclidean domain. 6

(d) Show that in a principal ideal domain, every non-zero prime ideal is maximal. 4

UNIT—V

9. Answer any *two* of the following questions : 2×2=4

(a) Give an example of a UFD which is not a PID.

(b) State Eisenstein's criterion.

(c) Show that $\frac{Q[x]}{I}$, where $I = \langle x^2 - 5x + 6 \rangle$, is not a field.

10. Answer *either* (a) and (b) *or* (c) and (d) :

(a) Prove that an integral domain R with unity is a UFD if and only if every non-zero, non-unit element is a finite product of primes. 6

(b) If R is an integral domain with unity and a is an irreducible element of R, then prove that a is irreducible element of $R[x]$. 4

(c) If F is a field, then prove that an ideal $\langle P(x)\rangle \neq \{0\}$ in $F[x]$ is maximal if and only if $P(x)$ is irreducible in $F[x]$. 4

(d) If R is a UFD, then prove that $R[x]$ is also a UFD. 6

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