**2024/TDC (CBCS)/EVEN/SEM/
MTMHCC-201T/229**

TDC (CBCS) Even Semester Exam., 2024

MATHEMATICS

(2nd Semester)

Course No. : MTMHCC-201T

(Real Analysis)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—I

1. Answer any *two* of the following questions : 2×2=4

(a) Define infimum of a set. Find the infimum of the following set :

$$\left\{1+\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$$

(b) Show that if B is countable subset of an uncountable set A, then $A-B$ is uncountable.

(c) State the completeness property of $\mathbb{R}$.

2. Answer any *one* of the following questions : 10

(a) (i) State and prove the Archimedean property of real numbers. 5

(ii) If A and B are countable sets, then prove that $A \times B$ is countable. 5

(b) (i) If x and y are any real numbers with $x < y$, then prove that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$. 5

(ii) Define a bounded subset of $\mathbb{R}$. If A and B are bounded subsets of $\mathbb{R}$, then prove that $A \cap B$ and $A \cup B$ are also bounded. 1+2+2=5

UNIT—II

3. Answer any *two* of the following questions : 2×2=4

(a) Prove that a finite set has no limit point.

(b) Give an example to show that an arbitrary union of closed sets may not be closed.

(c) Show that the set

$$\left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \cdots\right\}$$

is neither open nor closed in $\mathbb{R}$.

4. Answer any *one* of the following questions : 10

(a) (i) Prove that the derived set of a set is closed. 5

(ii) Show that every open set in $\mathbb{R}$ is a union of open intervals.

(b) (i) State and prove Bolzano-Weierstrass theorem for sets. 1+5=6

(ii) If A and B are subsets of the set of real numbers, then prove that—

(1) $A \subseteq B \Rightarrow D(A) \subseteq D(B)$

(2) $D(A \cup B) = D(A) \cup D(B)$

where $D(F)$ denotes the derived set of $F \subseteq \mathbb{R}$. 2+2=4

UNIT—III

5. Answer any *two* of the following questions : 2×2=4

(a) Show that the sequence $\left\langle \frac{1}{3^n} \right\rangle$ converges to zero.

(b) Define a bounded sequence in $\mathbb{R}$. Give an example of a sequence which is neither bounded above nor bounded below.

(c) Give an example of two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ in $\mathbb{R}$ such that $\langle x_n \rangle$ and $\langle y_n \rangle$ are non-convergent but—

(i) their sum $\langle x_n + y_n \rangle$ converges;

(ii) their product $\langle x_n y_n \rangle$ converges.

6. Answer any *one* of the following questions : 10

(a) (i) Prove that every convergent sequence of real numbers is bounded. Give an example to show the converse of the above result is not true. 3+2=5

(ii) Show that the sequence $\langle x_n \rangle$ defined by $x_n = \sqrt{n+1} - \sqrt{n}$, $\forall\, n \in \mathbb{N}$ is convergent. 5

(b) (i) State and prove monotone convergence theorem. 1+5=6

(ii) Prove that a sequence in $\mathbb{R}$ can have at most one limit. 4

UNIT—IV

7. Answer any *two* of the following questions : 2×2=4

(a) Prove that the sequence $\{\frac{1}{n}\}$ is a Cauchy sequence.

(b) Give an example of an unbounded sequence that has a convergent subsequence.

(c) State monotone subsequence theorem.

8. Answer any *one* of the following questions : 10

(a) (i) State and prove Bolzano-Weierstrass theorem for sequences. 1+5=6

(ii) Define a Cauchy sequence in $\mathbb{R}$. Prove that the sequence (n) is not a Cauchy sequence in $\mathbb{R}$. 1+3=4

(b) (i) Prove that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. 6

(ii) Show that the sequence (x_n), where

$$x_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$$

cannot converge. 4

UNIT—V

9. Answer any *two* of the following questions : 2×2=4

(a) Show that the series

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$$

is not convergent.

(b) State Leibnitz test.

(c) Define conditional convergent series. Give an example of it.

10. Answer any *one* of the following questions : 10

(a) *(i)* If a series in $\mathbb{R}$ is absolutely convergent, then prove that it is convergent. 4

(ii) Test for convergence of the following series : 3×2=6

(1) $\frac{\alpha}{\beta}+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\cdots$

(2) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\cdots$

(b) *(i)* Prove that the positive termed geometric series $1+r+r^2+\cdots$ converges for $r<1$, and diverges to $+\infty$ for $r\geq 1$. 5

(ii) Prove that the series

$$\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p},\ p>0$$

converges for $p>1$ and diverges for $p\leq 1$. 5

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