2023/TDC(CBCS)/ODD/SEM/
MTMDSC/GE–301T/308

TDC (CBCS) Odd Semester Exam., 2023

MATHEMATICS

(3rd Semester)

Course No. : MTMDSC/GE–301T

(Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer *twenty* questions, selecting *four* from each Unit : 1×20=20

UNIT—I

1. Define infimum of a set.

2. Find lower bound and upper bound of the set

$$\left\{1+\frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$$

3. Write the completeness property of $\mathbb{R}$.

4. Give an example of an uncountable set.

5. State True or False :

"Subset of an unbounded set is unbounded."

UNIT—II

6. Define open set.

7. Give an example of sequence of nested intervals.

8. Write the cluster point of the set

$$\left\{1-\frac{1}{n} : n \in \mathbb{N}\right\}$$

9. Give an example of a set which is neither open nor closed.

10. State True or False :

"A subset of $\mathbb{R}$ is closed if and only if it contains all of its cluster points."

UNIT—III

11. Define bounded sequence.

12. Give an example of a sequence which is bounded but not convergent.

13. Under what condition a bounded sequence is convergent.

14. Define monotonic sequence.

15. Define Cauchy sequence.

UNIT—IV

16. Write the sum of infinite geometric series whose first term is a and common ratio is r, $|r|<1$.

17. State Cauchy's criterion for series.

18. State D'Alembert's ratio test.

19. Define alternating series.

20. Give an example of an absolutely convergent series.

UNIT—V

21. State sequential criterion for continuity of a function at a point.

22. Does

$$\lim_{x \to 0} x \sin\left(\frac{1}{x^2}\right)$$

exist in $\mathbb{R}$?

23. State Cauchy's criterion for finite limit of a function at a point.

24. Give an example of a function defined on $\mathbb{R}$ which is continuous only at $x = 0$.

25. State True or False :

"If a function f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then it assumes every value between $f(a)$ and $f(b)$."

SECTION—B

Answer *five* questions, selecting *one* from each Unit : 2×5=10

UNIT—I

26. Prove that subset of a bounded set is bounded.

27. If

$$S := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$$

then prove that infimum of S is 0.

UNIT—II

28. Write nested interval theorem.

29. Give an example to show that an arbitrary intersection of open sets may not be open.

UNIT—III

30. Using ε-δ definition, show that

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

31. Prove that the sequence $\langle \frac{1}{n^2} \rangle$ is monotonically decreasing sequence.

UNIT—IV

32. Find the sum of the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

33. Show that the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

is not convergent.

UNIT—V

34. Show that the function defined as

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$$

has removable discontinuity at the origin.

35. Examine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at $x = 0$.

SECTION—C

Answer *five* questions, selecting *one* from each Unit : 8×5=40

UNIT—I

36. *(a)* Prove that the set of rational numbers is countable. 5

(b) If $a \in \mathbb{R}$ be such that $0 \le a < \varepsilon$ for every $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$, then prove that $a = 0$. 3

37. (a) Find upper bound and lower bound of the set $\{x \in \mathbb{R} : x^2 - 3x + 2 < 0\}$. 4

(b) Prove that for any $\varepsilon > 0$ there exists a positive integer n such that $\frac{1}{n} < \varepsilon$ (by using Archimedean property). 4

UNIT—II

38. (a) Show that every open set in $\mathbb{R}$ is a union of open intervals. 4

(b) Prove that a set is closed if and only if its complement is open. 4

39. (a) Prove that the derived set of a set is closed. 4

(b) If S, T are subsets of $\mathbb{R}$, then show that $D(S \cap T) \subseteq D(S) \cap D(T)$. 2

(c) Give an example to show that $D(S \cap T)$ and $D(S) \cap D(T)$ may not be equal. 2

UNIT—III

40. (a) Prove that limit of a sequence is unique if it exists. 3

(b) Write squeeze theorem. By using squeeze theorem, prove that

$$x_n = \frac{\sin n}{n}$$

is a convergent sequence, where $n \in \mathbb{N}$.

2+3=5

41. (a) Write monotone convergence theorem and by using this, prove that the sequence (x_n) is convergent and find its limit, where x_n is defined as follows : 1+5=6

$$x_1 = 8 \text{ and } x_{n+1} = \frac{1}{2}x_n + 2 \quad \text{for } n \in \mathbb{N}$$

(b) Prove that the sequence

$$x_n = 1 + \frac{1}{n}, \; n \in \mathbb{N}$$

is a Cauchy sequence. 2

UNIT—IV

42. Test for convergence of the following series : 4+4=8

(i) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$

(ii) $\sum_{n=0}^{\infty} \frac{\lfloor n}{5^n}$

43. (a) State root test for convergence of a series and by using root test, determine if the following series is convergent or divergent : 1+4=5

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

(b) Give an example of a series which is conditionally convergent but not absolutely convergent. 3

UNIT—V

44\. (a) Prove that the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational} \end{cases}$$

is discontinuous at every point. 4

(b) Prove that if a function is continuous in a closed and bounded interval, then it is bounded therein. 4

45\. (a) Prove that a function f defined on an interval I is continuous at a point $c \in I$ if an only if for every sequence $\{c_n\}$ in I converging to c, we have

$$\lim_{n \to \infty} f(c_n) = f(c)$$ 5

(b) Examine the following function for continuity at the origin : 3

$$f(x) = \begin{cases} \dfrac{xe^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

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