2023/TDC(CBCS)/ODD/SEM/
MTMHCC–302T/306

TDC (CBCS) Odd Semester Exam., 2023

MATHEMATICS

(Honours)

(3rd Semester)

Course No. : MTMHCC–302T

(Group Theory)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer *ten* questions, selecting any *two* from each Unit : 2×10=20

UNIT—I

1. Show that the identity element in a group is unique.

2. Construct the Cayley table for the group of 4th roots of unity with respect to multiplication.

3. In a group G, show that $(ab)^{-1} = b^{-1}a^{-1}$ for every $a, b \in G$.

UNIT—II

4. Justify that the set of irrational numbers is a subgroup of the group of non-zero real numbers under multiplication.

5. Define centralizer of an element in a group. Give an example of a centralizer.

6. Give example to show that the union of two subgroups need not be a subgroup.

UNIT—III

7. Show that every cyclic group is Abelian.

8. Determine whether the following permutations are even or odd :

(a) (1 3 5 6)

(b) (1 2 4 3) (3 5 2)

9. Justify that S_{2023}, i.e., the symmetric group of degree 2023 is Abelian.

UNIT—IV

10. Let H be a subgroup of a group G and $a \in H$. Show that $aH = H$.

11. What are the subgroups of a group of order 47?

12. Define factor group. What is the identity element in such a group?

UNIT—V

13. Let ϕ be a homomorphism from a group G to a group $\overline{G}$. Show that ϕ carries identity of G to the identity of $\overline{G}$.

14. Can there be an isomorphism from the group $(\mathbb{Z}, +)$ of integers to the group $(\mathbb{R}, +)$ of real numbers? Justify.

15. Give example of a homomorphism from the group $(\mathbb{R}, +)$ to the multiplicative group $(\mathbb{R}, \cdot)$ of non-zero real numbers. Justify your answer.

SECTION—B

Answer *five* questions, selecting *one* from each Unit : 10×5=50

UNIT—I

16. *(a)* Show that the set $GL(2, \mathbb{R})$ of 2×2 matrices over $\mathbb{R}$ with non-zero determinant is a group w.r.t. matrix multiplication. Is it Abelian? Justify your answer. 5+1=6

(b) Show that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ $\forall$ $a, b \in G$. 4

17. (a) Let $U(n)$ be the set of all positive integers less than n and relatively prime to n. Construct the Cayley table for $U(10)$ w.r.t. multiplication modulo 10. Is $U(10)$ a group? What are the elements in $U(n)$ if n is a prime? 4+1=5

(b) Show that the left and right cancellation laws hold in a group. Hence show that inverse of each element in a group is unique. 3+2=5

UNIT—II

18. (a) Let G be a group and H be a non-empty subset of G. Show that H is a subgroup of G if and only if $ab \in H$ $\forall$ $a, b \in H$ and $a^{-1} \in H$ $\forall$ $a \in H$. 5

(b) Show that the center of a group G is a subgroup of G. 5

19. (a) Let $\mathbb{C}^*$ be the multiplicative group of all non-zero complex numbers. Let $H = \{ z \in \mathbb{C}^* \mid |z| = 1\}$. Prove that H is a subgroup of $\mathbb{C}^*$. 5

(b) Show that the intersection of a finite number of subgroups of a group is also a subgroup of that group. 5

UNIT—III

20. (a) Let G be a group and $a \in G$ be an element of order n. If $a^m = e$, the identity in G, then show that n divides m. 5

(b) Show that every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. 5

21. (a) If a is a generator of a cyclic group G of order n, then show that a^k is a generator of G if and only if g.c.d. $(k, n) = 1$. 5

(b) Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{pmatrix} \text{ and}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

Find α^{-1}, $\beta\alpha$ and $\alpha\beta$. 5

UNIT—IV

22. *(a)* Let H be a subgroup of a group G and let $a, b \in G$. Show that either $aH \cap bH = \phi$ or $aH = bH$. 5

(b) Show that a subgroup H of a group G is normal if and only if $xHx^{-1} \subseteq H$ for all x in G. 5

23. *(a)* Show that $G \oplus H$ is Abelian if and only if G and H are Abelian. Here, $G \oplus H$ is the external direct product of G and H. 5

(b) Let G be a group and H be a normal subgroup of G. Consider $\frac{G}{H} = \{aH \mid a \in G\}$ under the operation $(aH)(bH) = (ab)H$. Show that $\frac{G}{H}$ is a group w.r.t. this operation. 5

UNIT—V

24. *(a)* Let ϕ be a homomorphism from a group G to a group $\overline{G}$. If ϕ is onto and $\ker \phi = \{e\}$, then show that ϕ is an isomorphism. 5

(b) If K is a subgroup of G and N is a normal subgroup of G, then prove that $K/(K \cap N)$ is isomorphic to KN/N. 5

25. *(a)* Let ϕ be an isomorphism from a group G to a group $\overline{G}$. If K is a subgroup of G, then show that $\phi(K) = \{\phi(k) \mid k \in K\}$ is a subgroup of $\overline{G}$. 5

(b) State and prove Cayley's theorem. 5

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