2023/TDC(CBCS)/ODD/SEM/
MTMDSC/GE–101T/304

TDC (CBCS) Odd Semester Exam., 2023

MATHEMATICS

(1st Semester)

Course No. : MTMDSC/GE–101T

(Differential Calculus)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer *twenty* questions, selecting *four* from each Unit : 1×20=20

UNIT—I

1. Define ε-δ definition of limit of a function.

2. What is the value of $\lim_{x \to 0} x \sin \frac{1}{x}$?

3. Does $\lim_{x \to 0} \frac{|x|}{x}$ exist?

4. What is the value of $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$?

5. Consider the function

$$f(x) = \begin{cases} x+1, & x \le 3 \\ x^3, & x > 3 \end{cases}$$

Does $\lim_{x \to 3} f(x)$ exist?

UNIT—II

6. A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} \frac{1}{2} - x, & \text{when } 0 < x < \frac{1}{2} \\ \frac{1}{2}, & \text{when } x = \frac{1}{2} \\ \frac{3}{2} - x, & \text{when } \frac{1}{2} < x < 1 \end{cases}$$

Is $f(x)$ continuous at $x = \frac{1}{2}$?

7. A function is continuous throughout a closed interval. Is it bounded therein?

8. Find $\frac{d}{dx}(x^{1/2} + x \log x + 3 \sin^{-1} x)$.

9. Give an example of a function which is continuous but not differentiable.

10. Find the points where the function $f(x) = \frac{1}{\log|x|}$ is discontinuous.

UNIT—III

11. Find y_n, when $y = \sin(ax+b)$.

12. State Leibnitz's theorem.

13. If $y = a\cos(\log x) + b\sin(\log x)$, then show that

$$x^2 y_2 + x y_1 + y = 0$$

14. State Euler's theorem on homogeneous function in two variables.

15. If $u = x^3 + x^2 y^2 + y^3$, then find $\frac{\partial^2 u}{\partial x \partial y}$.

UNIT—IV

16. Write the equation of the tangent to the curve $y = f(x)$ at $x = a$.

17. Find the length of the Cartesian subtangent of the curve $y = e^{-x/2}$.

18. Find the radius of curvature of $S = a(e^{m\psi} - 1)$, at any position ψ.

19. What is the gradient of the normal to the curve $y^2 = 4x$ at $(1, 2)$?

20. What is the angle between the two curves $y = x^2$ and $y^2 = x$ intersecting at a point $(1, 1)$?

UNIT—V

21. State Rolle's theorem.

22. State True or False :

"$f(c)$ is an extreme value if and only if $f'(c) = 0$."

23. Evaluate $\lim_{x \to \infty} \frac{x^4}{e^x}$.

24. Write $\sin x$ in ascending powers of x.

25. Write down Lagrange's form of remainder in Taylor's theorem.

SECTION—B

Answer *five* questions, selecting *one* from each Unit : 2×5=10

UNIT—I

26. Show that $\lim_{x \to 0} e^{1/x}$ does not exist.

27. Evaluate $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$.

UNIT—II

28. Let $f(x+y) = f(x) + f(y)$ for all x, y and $f'(0)$ exist. Prove that $f'(x) = f'(0)$ for all $x \in R$.

29. Find $f(0)$ so that $f(x) = x \sin \frac{\pi}{x}$ for $x \neq 0$ may be continuous at $x = 0$.

UNIT—III

30. If $y = \frac{x}{x+1}$, then show that $y_5(0) = \underline{|5}$.

31. If $f(x, y) = x^3 y + e^x y^2$, show that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

UNIT—IV

32. Find the point on the curve $y = x^3 - 6x + 7$ when the tangent is parallel to the straight line $y = 6x + 1$.

33. Find the length of the polar subtangent for the curve $r = a(1 + \cos\theta)$ at $\theta = \frac{\pi}{2}$.

UNIT—V

34. Evaluate $\lim_{x\to 1}\dfrac{\log(1-x)}{\cot(\pi x)}$.

35. Examine whether Lagrange's mean value theorem can be applied to the function $f(x)=|x|$ in the interval $[-1, 1]$.

SECTION—C

Answer *five* questions, selecting *one* from each Unit : 8×5=40

UNIT—I

36. *(a)* If $\lim_{x\to a} f(x)=l$, then prove that

$$\lim_{x\to a}|f(x)|=|l|$$

Give an example to show that the converse of the above result may not be true. 3+2=5

(b) Show that

$$\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}=\frac{2}{\pi}$$ 3

37. *(a)* Prove that if $\lim_{x\to a} f(x)$ exists, then the limit is unique. 4

(b) Find the values of the following : 2×2=4

(i) $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$

(ii) $\lim_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{\tan x}\right)$

UNIT—II

38. *(a)* Show that the function

$$f(x)=|x|+|x-1|+|x-2|$$

is continuous at the point $x=0, 1, 2$. 3

(b) Show that the function

$$f(x)=\begin{cases} x^2\cos(\frac{1}{x}), & x\neq 0\\ 0, & x=0\end{cases}$$

is continuous at $x=0$, but has no derivative there. 5

39. *(a)* Find the values of a and b such that the function

$$f(x)=\begin{cases} x+\sqrt{2}\,a\sin x, & 0\le x\le \frac{\pi}{4}\\ 2x\cot x+b, & \frac{\pi}{4}\le x\le\frac{\pi}{2}\\ a\cos 2x-b\sin x, & \frac{\pi}{2}\le x\le \pi\end{cases}$$

is continuous for all values of x in the interval $0\le x\le \pi$. 4

(b) If

$$f(x) = \begin{cases} x &, \quad 0 < x < 1 \\ 2 - x &, \quad 1 \le x \le 2 \\ x - \frac{1}{2}x^2 &, \quad x > 2 \end{cases}$$

then show that $f(x)$ is continuous at $x = 1$ and $x = 2$, and that $f'(2)$ exists, but $f'(1)$ does not. 4

UNIT—III

40. (a) If $y = e^{2\sin^{-1} x}$, then prove that

$$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + 4)y_n = 0$$

Also show that $(y_7)_0 = 2 \cdot 5 \cdot 13 \cdot 29$. 3+1=4

(b) If $u = \dfrac{x^2y^2}{x+y}$, apply Euler's theorem to find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ and hence deduce that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 6u$$ 4

41. (a) If $y = \dfrac{x^2 + 1}{(x-1)(x-2)(x-3)}$, then find y_n. 3

(b) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$ 3

(c) Verify Euler's theorem for the function

$$f(x) = ax^2 + 2hxy + by^2$$ 2

UNIT—IV

42. (a) If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the x-axis, then show that its equation is $y\cos\phi - x\sin\phi = a\cos 2\phi$. 4

(b) Find the radius of curvature of $y = xe^{-x}$ at its maximum point. 4

43. (a) For the parabola $y^2 = 4ax$, show that the subtangent is bisected at the vertex and that the subnormal is constant. 4

(b) Trace the curve $r = a(1 + \cos\theta)$. 4

UNIT—V

44. (a) State and prove Lagrange's mean value theorem. 4

(b) Show that of all rectangles of given area, the square has the smallest perimeter. 4

45. *(a)* Expand $\log_e(1+x)$ in Maclaurin's infinite series in power of x. 4

(b) Verify Rolle's theorem for the function $f(x) = x^3(x-1)^2$ in the interval $[0, 1]$. 4

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