2021/TDC (CBCS)/EVEN/SEM/ MTMHCC–601T/126

TDC (CBCS) Even Semester Exam., September—2021

MATHEMATICS

(6th Semester)

Course No. : MTMHCC–601T

(Complex Analysis)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* of the following questions : 2×10=20

1. Show that the statements $\text{Re}\{z\} > 0$ and $|z-1| < |z+1|$ are equivalent, $z \in \mathbb{C}$.

2. If the product of two complex numbers z_1 and z_2 is a non-zero real number, prove that there exists a real number γ such that

$$z_1 = \gamma \bar{z}_2$$

3. If $z = Re^{i\theta}$, find the value of $|e^{iz}|$.

4. Find the locus of the point $z = x + iy$ satisfying the equation

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$$

5. Write down the polar form of Cauchy-Riemann equations.

6. Show that the function $f(z) = xy + iy$ is everywhere continuous but not analytic.

7. Show that $u = y^3 - 3x^2 y$ is a harmonic function. Determine its harmonic conjugate.

8. Show that the function $f : \mathbb{C} \to \mathbb{R}$ defined by $f(z) = |z|$ is nowhere differentiable.

9. Write the statement of Cauchy's theorem.

10. Define simply and multi-connected regions.

11. Evaluate

$$\int_C \frac{e^{2z}}{(z+1)^4} dz$$

where C is $|z| = 3$.

12. Evaluate by Cauchy's integral formula

$$\int_C \frac{dz}{z(z-\pi i)}$$

where C is $|z+3i|=1$.

13. Define entire function with examples.

14. State Morera's theorem. Does this theorem applicable in a multi-connected region?

15. Does the series

$$z(1-z)+z^2(1-z)+z^3(1-z)+\cdots$$

converges for $|z|<1$? Explain.

16. Under what conditions the series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$$

converges and diverges?

17. What is meromorphic function?

18. Specify the nature of singularity at $z=a$ of

$$f(z)=\frac{\cot \pi z}{(z-a)^2}$$

19. Find the poles of

$$\left(\frac{z+1}{z^2+1}\right)^2$$

20. Find the residues of

$$f(z)=\frac{e^z}{z^2(z^2+9)}$$

SECTION—B

Answer any *five* of the following questions : 10×5=50

21. *(a)* Prove that

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$$

interpret the result geometrically and hence deduce that

$$|z_1+\sqrt{z_1^2-z_2^2}|+|z_1-\sqrt{z_1^2-z_2^2}|$$
$$=|z_1+z_2|+|z_1-z_2|,\ z_1, z_2\in\mathbf{C}$$ 5

(b) Show that the origin and the points representing the roots of the equation $z^2+pz+q=0$ form an equilateral triangle if $p^2=3q$. 5

22. (a) Show that the equation of a straight line in the Argand plane can be put in the form

$$z\bar{b} + b\bar{z} + c = 0$$

where $b \neq 0 \in \mathbb{C}$ and $c \in \mathbb{R}$. 5

(b) Find all circles in the complex plane which are orthogonal to

$$|z| = 1 \text{ and } |z-1| = 4$$ 5

23. (a) Prove that a necessary and sufficient condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathbb{R}$ is that the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ are satisfied in $\mathbb{R}$, where it is supposed that these partial derivatives are continuous in $\mathbb{R}$. 7

(b) If u and v are harmonic in a region $\mathbb{R}$, prove that

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

is analytic in $\mathbb{R}$. 3

24. (a) Show that the function $f(z) = \sqrt{|xy|}$ is not regular at the origin although the Cauchy-Riemann equations are satisfied at the origin. 4

(b) If $w = f(z) = u(x, y) + iv(x, y)$ is an analytic function of $z = x + iy$ and

$$u - v = e^x(\cos y - \sin y)$$

find w in terms of z. 3

(c) Prove that the function

$$u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

satisfies Laplace equation and determine the corresponding analytic function. 3

25. (a) State and prove Cauchy's integral formula. 6

(b) Evaluate

$$\int_C \frac{dz}{z - a}$$

where C is any closed curve and $z = a$ is

(i) outside C;

(ii) inside C. 4

26. (a) Using the definition of an integral as the limit of a sum, evaluate the following integrals :

(i) $\int_L dz$

(ii) $\int_L |dz|$

where L is any rectifiable arc joining the points $z = \alpha$ and $z = \beta$. 4

(b) Evaluate

$$\int_L \frac{dz}{z-a}$$

where L represents a circle $|z-a|=r$. 2

(c) Prove that if $f(z)$ is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \leq M$ on C, then

$$\left|\int_C f(z)\,dz\right| \leq ML$$ 4

27. (a) State and prove Liouville's theorem. 6

(b) Prove that every polynomial equation

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$$

where the degree $n \geq 1$ and $a_n \neq 0$, has exactly n roots. 4

28. (a) State and prove the fundamental theorem of algebra. 6

(b) Obtain the Taylor series which represents the function

$$\frac{z^2-1}{(z+2)(z+3)}$$

in the region $|z|<2$. 4

29. *(a)* Define different types of singularities with examples. 5

(b) State and prove Cauchy's residue theorem. 5

30. Evaluate the following integrations by Cauchy's residue theorem (any *two*) : 5×2=10

(i) $\oint_C \frac{dz}{z^3+1}$, where C is the circle defined by $|z|=2$

(ii) $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ if $a>|b|$

(iii) $\oint_C \frac{e^z}{(z^2+\pi^2)^2} dz$, where $C: |z|=4$

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