**TDC (CBCS) Even Semester Exam.,
September—2021**

MATHEMATICS

(6th Semester)

Course No. : MTMHCC-602T

(Linear Algebra)

Full Marks : 70
Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions

SECTION—A

Answer any *ten* questions from Q. Nos. **1** to **20** :

2×10=20

1. Justify whether the set

$$W = \{(x, y, 2) \mid x, y \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^3(\mathbb{R})$.

2. Define linear dependence and independence of vectors in a vector space.

3. Check if the set $S = \{(1, 0), (1, 2)\}$ is a basis of $\mathbb{R}^2(\mathbb{R})$.

4. Give example to justify that union of two subspaces of a vector space need not be a subspace.

5. Define linear transformation from a vector space U to a vector space V.

6. Find the null space of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x, x+y, x+y+z)$$

7. Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (y, x)$ with respect to the standard ordered basis of $\mathbb{R}^2(\mathbb{R})$.

8. Give example of a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is not a linear transformation.

9. Justify if $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x, y, 0)$ is an isomorphism.

10. If U and V are vector spaces and $T : U \to V$ is a one-one linear transformation, then what is the null space of T? Justify your answer.

11. Let $T: U \to V$ be an isomorphism and $S = \{u_1, u_2, \cdots, u_n\}$ be a linearly independent set in U. Justify if $T(S)$ is linearly independent in V.

12. Let $T: U \to V$ be a linear transformation and $C \in \mathbb{R}$ be a scalar. Define the linear transformation CT and justify that it is a linear transformation.

13. Define eigenvalue of a linear operator. Give an example.

14. Let $T: V \to V$ be a linear operator. When is a subspace W of V said to be invariant under T?

15. If v is an eigenvector of $T: V \to V$ corresponding to the eigenvalue λ and $\alpha \in \mathbb{R}$ be a non-zero scalar, then show that αv is also an eigenvector of T corresponding to the eigenvalue λ.

16. Write the characteristic polynomial of

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and find its eigenvalues.

17. Define an inner product space.

18. State Cauchy-Schwartz inequality. Comment on the case when the equality holds.

19. If x, y are orthogonal to each other in an inner product space V, show that

$$||x+y||^2 = ||x||^2 + ||y||^2$$

20. Define orthogonal complement of a set in an inner product space.

SECTION—B

Answer any *five* questions from Q. Nos. **21** to **30** : 10×5=50

21. *(a)* Define a vector space over a field F. Show that a vector space has a unique additive identity. If $\overline{0}$ is the additive identity (or zero vector) in a vector space $V(F)$, then show that $\alpha \cdot \overline{0} = \overline{0}\ \forall \alpha \in F$. 2+1+2=5

(b) Show that a non-empty subset W of a vector space V is a subspace of V if and only if it is closed under vector addition and scalar multiplication. 5

22. *(a)* Show that the intersection of any family of subspaces of a vector space is a subspace. 5

(b) Let V be a finite dimensional vector space. Show that any two bases of V have the same number of elements. 5

23. (a) Let $T: U \to V$ be a linear transformation. Define null space of T and show that it is a subspace of U. 1+3=4

(b) State and prove the Rank-Nullity theorem for finite dimensional vector spaces. 1+5=6

24. (a) Consider the ordered bases

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

and $$\overline{B} = \{(0, 0, 1), (1, 0, 0), (0, 1, 0)\}$$

of $\mathbb{R}^3(\mathbb{R})$. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x+y, y+z, z+x)$$

Find the matrix of T w.r.t. the ordered bases B and $\overline{B}$. Also, find the matrix of T w.r.t. $\overline{B}$ and B. 2½+2½=5

(b) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(x, y) = (x, x+y, y)$$

Find the range space and null space of T. Hence find the rank and nullity of T.

2+1+2=5

25. *(a)* Let V and W be any two vector spaces. Show that the set $L(V, W)$ of all linear transformations from V to W is a vector space with the operations of addition $(S+T)$ and scalar multiplication (αS) defined as

$$(S+T)(x) = S(x) + T(x)$$
$$(\alpha S)(x) = \alpha \cdot S(x)$$

for all $S, T \in L(V, W)$ and $\alpha \in \mathbb{R}$. 6

(b) Let V be a finite dimensional vector space. Show that a linear map $T: V \to V$ is an isomorphism if and only if $\ker T = \{0\}$, where $\ker T$ is the null space of T. 4

26. *(a)* Let U and V be vector spaces and $T: U \to V$ be a linear transformation. Then prove that

$$R(T) \cong U / N(T)$$

where $R(T)$ and $N(T)$ are the range space and null space of T respectively. 5

(b) Show that a linear transformation $T: V \to W$ is invertible iff it is bijective. 5

27. *(a)* Let $T: V \to V$ be a linear map and λ be an eigenvalue of T. Show that the set

$$E = \{v \in V \mid T(v) = \lambda v\}$$

is a subspace of V. Does the space E consist entirely of eigenvectors of T w.r.t. the eigenvalue λ? Justify. 3+1=4

(b) Verify Cayley-Hamilton theorem for the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Hence find A^{-1}. 4+2=6

28. *(a)* Let $T: V \to W$ be a linear map. Show that the range space and null space of T are invariant under T. 2+2=4

(b) State and prove Cayley-Hamilton theorem for a square matrix. 1+5=6

29. *(a)* Show that $\mathbb{R}^n (\mathbb{R})$ is an inner product space with inner product defined as for

$$x = (x_1, x_2, \cdots, x_n)$$

and $$y = (y_1, y_2, \cdots, y_n)$$

in $\mathbb{R}^n$, then

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$ 5

(b) Let W be a subset of an inner product space V. Show that the orthogonal complement of W is a subspace of V. 5

30. (a) State and prove Bessel's inequality. 1+5=6

(b) Let V be a real inner product space. Then show that

$$||x+y||^2 - ||x-y||^2 = 4\langle x, y\rangle$$

for all $x, y \in V$. 4

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